JHEP HYPER VERSION

Received: *June 11, 2001*, Accepted: *October 8 2001*

The operator algebra and twisted KZ equations of WZW orbifolds

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JHEP10(2001)011

Abstract: We obtain the operator algebra of each twisted sector of all WZW orbifolds, including the general twisted current algebra and the algebra of the twisted currents with the twisted affine primary fields. Surprisingly, the twisted right and left mover current algebras are not a priori copies of each other. Using the operator algebra we also derive world-sheet differential equations for the twisted affine primary fields of all WZW orbifolds. Finally we include ground state properties to obtain the twisted Knizhnik-Zamolodchikov equations of the WZW permutation orbifolds and the inner-automorphic WZW orbifolds.

Keywords: Conformal Field Models in String Theory, Conformal and W Symmetry, Discrete and Finite Symmetries.

Contents

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1. Introduction

Affine Lie algebra [1]–[4], or current algebra on the circle, is the basis of a very large set of conformal field theories called the affine-Virasoro constructions [4]–[7] A. The affine-Virasoro constructions include as important special cases the affine-Sugawara constructions [3], [8]–[11] and [4] A_g on Lie g, the coset constructions [3, 8, 12, 4] $A_{g/h}$ and the H-invariant conformal field theories [13, 4] $A(H)$, where $H \subset \mathrm{Aut}(g)$ is any finite symmetry group.

Orbifold theory [14]–[30] has historically been approached at the level of examples, but recently a construction was given for the left mover stress tensors of all sectors of the *general current-algebraic orbifold* [31, 32]

$$\frac{A(H)}{H}, \qquad H \in \mathrm{Aut}(g)\,, \tag{1.1}$$

where $A(H)$ is any H-invariant CFT. This construction drew heavily on recent advances [33, 34] in the theory of cyclic permutation orbifolds, and the construction has been worked out in further detail for the general cyclic permutation orbifold [31, 32], the general S_N permutation orbifold [32], the general inner-automorphic orbifold [32], the general WZW orbifold [31, 32], the general coset orbifold [32] and the cyclic coset orbifolds [35].

The principles of this construction are given in refs. [31] and [32]:

• The *H-eigenvalue problem*, whose eigendata encodes the action of H on the untwisted currents (the action of H in the adjoint of g).

• *Eigencurrents*, constructed from the eigendata of the H-eigenvalue problem, with diagonal response to transformations in the symmetry group.

• *Local isomorphisms*, which provide a map from the eigencurrents and untwisted stress tensor to the *twisted currents* $\hat{J}(\sigma)$ with definite monodromy and the *orbifold stress tensor* $\hat{T}_\sigma$

$$\hat{T}_\sigma(z) = \mathcal{L}^{n(r)\mu;-n(r),\nu}(\sigma) : \hat{J}_{n(r)\mu}(z,\sigma)\hat{J}_{-n(r),\nu}(z,\sigma) :, \qquad \sigma = 0,\ldots,N_c - 1$$

$$\hat{J}_{n(r)\mu}(ze^{2\pi i},\sigma) = e^{-2\pi i \frac{n(r)}{\rho(\sigma)}} \hat{J}_{n(r)\mu}(z,\sigma), \qquad \hat{T}_\sigma(ze^{2\pi i}) = \hat{T}_\sigma(z) \tag{1.2}$$

in each sector σ of the orbifold $A(H)/H$. Here N_c is the number of conjugacy classes of H, the σ-dependent integers $\rho(\sigma)$ and $n(r)$ are included in the eigendata of the H-eigenvalue problem and $\mathcal{L}(\sigma)$ is the *twisted inverse inertia tensor* of sector σ.

The twisted inverse inertia tensor is an example of an *orbifold duality transformation* [31, 35, 32]

$$L_H \underset{\sigma}{\longrightarrow} \mathcal{L}(L_H,\sigma) \tag{1.3}$$

which expresses the twisted tensor in each sector of the orbifold as a discrete Fourier transform of the H-invariant inverse inertia tensor L_H of the symmetric theory $A(H)$. Other duality transformations include the *twisted structure constants* and the *twisted metric* of the general twisted left mover current algebra [31, 32].

In this paper, we restrict our attention to the special case of the *general WZW orbifold*

$$\frac{A_g(H)}{H} \subset \frac{A(H)}{H}, \qquad H \subset \mathrm{Aut}(g) \tag{1.4}$$

while extending the construction to obtain the full operator algebra of these orbifolds. In particular (see section 5), we obtain for each sector σ of each $A_g(H)/H$:

• the twisted right and left mover current algebra, and the corresponding right and left mover twisted affine-Sugawara constructions,

• the algebra of the twisted currents and twisted affine-Sugawara constructions with the *twisted affine primary fields*,

• the WZW orbifold operator equations of motion, which are world-sheet differential equations for the twisted affine primary fields.

In this development we encounter some natural extensions of the principles above, including:

• the *extended H-eigenvalue problem* (see subsection 3.2), which encodes the action of H in any matrix representation T of g,

• *eigenfields* (see section 4), including right and left mover eigencurrents and now the eigenprimary fields, which are constructed from the affine primary fields and the eigendata of the extended H-eigenvalue problem.

• extended local isomorphisms from the eigenfields to the twisted fields with definite monodromy, including now the twisted right and left mover currents and the twisted affine primary fields.

The extended construction also includes a number of new duality transformations associated to the twisted affine primary fields. We mention in particular (see subsection 4.2) the *twisted representation matrices* $\mathcal{T}_{n(r)\mu} \equiv \mathcal{T}_{n(r)\mu}(T,\sigma)$,

$$T \underset{\sigma}{\longrightarrow} \mathcal{T}(T,\sigma) \tag{1.5}$$

which are the duality transformations in sector σ of the untwisted matrix irreps T of g.

One surprise in the full operator algebra is that *the twisted right and left mover current algebras are not a priori copies of each other* (see subsections 5.3, 5.4 and 5.5), a conclusion which we check both at the level of characters (see subsection 5.4) and by constructing the corresponding *classical theory of WZW orbifolds* (see subsection 5.7). For permutation orbifolds (see subsections 7.3, 7.4 and 7.5) and inner-automorphic orbifolds (see subsection 10.1) we are able to show by a mode relabelling that the situation is equivalent to copies, but we do not have a proof for the general case.

Finally, we include ground state conditions to obtain *twisted Knizhnik-Zamolodchikov (KZ) systems* for all sectors of the general WZW permutation orbifold (see section 8) and the general inner-automorphic WZW orbifold (see section 10). These systems are a central result of this paper.

As an example, we find the twisted KZ system

$$\frac{A_g(H)}{H}, \qquad H(\text{permutation}) \subset \text{Aut}(g) \tag{1.6}$$

$$\partial_\mu \hat{A}_+(\mathcal{T},z,\sigma) = \hat{A}_+(\mathcal{T},z,\sigma)\hat{W}_\mu(\mathcal{T},z,\sigma), \qquad \sigma = 0,\ldots,N_c-1 \tag{1.7}$$

$$\begin{aligned}\hat{W}_\mu(\mathcal{T},z,\sigma) &= 2\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)aj;-n(r),bl}(\sigma) \times \\ &\times \left[\sum_{\nu\neq\mu}\left(\frac{z_\nu}{z_\mu}\right)^{\frac{\bar{n}(r)}{\rho(\sigma)}}\frac{1}{z_{\mu\nu}}\mathcal{T}^{(\nu)}_{n(r)aj}\mathcal{T}^{(\mu)}_{-n(r),bl} - \frac{\bar{n}(r)}{\rho(\sigma)}\frac{1}{z_\mu}\mathcal{T}^{(\mu)}_{n(r)aj}\mathcal{T}^{(\mu)}_{-n(r),bl}\right]\end{aligned} \tag{1.8}$$

for the correlators of the twisted affine primary fields in each twisted left mover sector of all WZW permutation orbifolds. Here $\{\mathcal{T}_{n(r)aj}\}$ and $\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}(\sigma)$ are respectively the twisted representation matrices of sector σ and the twisted inverse inertia tensor of the twisted affine-Sugawara constructions of that sector. The integers $\bar{n}(r)$ are the pullback of the integers $n(r)$ into a fundamental range. Simple solutions of this system are discussed in section 9, and the corresponding twisted KZ system for the inner-automorphic WZW orbifolds is solved in subsection 10.5.

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2. The general semisimple WZW model

2.1 Notation

In the following subsections, we will review the general semisimple WZW model A_g in the composite notation

$$(T_a)_\alpha{}^\beta\,, \qquad [T_a, T_b] = if_{ab}{}^cT_c\,, \qquad M(k,T)_\alpha{}^\beta = \left(\frac{k}{y(T)}\right)_\alpha \delta_\alpha{}^\beta \tag{2.1}$$

$$\mathrm{Tr}(M(k,T)T_aT_b) = G_{ab}\,, \qquad G_{ac}G^{cb} = \delta_a{}^b\,, \qquad [M(k,T), T_a] = 0 \tag{2.2}$$

$$a, b, c = 1, \dots, \dim\, g\,, \qquad \alpha, \beta = 1, \dots, \dim\, T\,. \tag{2.3}$$

Here T_a is any matrix representation of the compact semisimple Lie algebra g, with metric G_{ab} and structure constants $f_{ab}{}^c$. The diagonal *data matrix* $M(k,T)$ stores information about the Dynkin indices $\{y\}$ of the matrix representations and the affine levels $\{k\}$ of each simple component of g. When needed, the composite notation can be replaced by the explicit notation

$$g = \oplus_I \mathfrak{g}^I\,, \qquad I = 0, \dots, K-1, \qquad a \longrightarrow a(I), I\,, \qquad \alpha \longrightarrow \alpha(I), I \tag{2.4}$$

$$G_{ab} \longrightarrow G_{aI,bJ} = k_I \eta^I_{a(I),b(I)}\delta_{IJ}, \qquad , \qquad f_{ab}{}^c \longrightarrow f_{aI,bJ}{}^{cK} = f^I_{a(I),b(I)}{}^{c(I)}\delta_{IJ}\delta_J{}^K \tag{2.5}$$

$$(T_a)_\alpha{}^\beta \quad \longrightarrow \quad (T_{aI})_{\alpha J}{}^{\beta K} = (T^I_a)_{\alpha(I)}{}^{\beta(I)}\delta_{IJ}\delta_J{}^K, \qquad \mathrm{Tr}(T^I_aT^J_b) = y_I(T^I)\eta^I_{a(I),b(I)}\delta^{IJ} \tag{2.6}$$

$$M(k,T)_\alpha{}^\beta \longrightarrow M(k,T)_{\alpha I}{}^{\beta J} = \frac{k_I}{y_I(T^I)}\delta_{\alpha(I)}{}^{\beta(I)}\delta_I{}^J\,. \tag{2.7}$$

Here I is the semisimplicity index and $a(I)$, $\alpha(I)$ are indices associated to the simple component $\mathfrak{g}^I$, with structure constants, Killing metric, matrix representation and affine level f^I, η^I, T^I and k_I respectively. Note that the data matrix is invertible and that the matrix representations T of g are direct sums of the matrix representations T^I of $\mathfrak{g}^I$.

We also need the left and right mover *affine-Sugawara constructions* [3], [8]–[11] and [4] on g

$$\begin{gathered}
T(z) = L^{ab}_g : J_a(z)J_b(z) :, \qquad \bar{T}(\bar{z}) = L^{ab}_g : \bar{J}_a(\bar{z})\bar{J}_b(\bar{z}) :, \qquad c_g = 2G_{ab}L^{ab}_g \\
J_a(z) \quad \longrightarrow \quad J_{aI}(z), \qquad \bar{J}_a(\bar{z}) \quad \longrightarrow \quad \bar{J}_{aI}(\bar{z}) \\
L^{ab}_g \quad \longrightarrow \quad L^{aI,bJ}_g = \frac{\eta^{ab}_I\delta^{IJ}}{2k_I + Q_I}, \qquad D_g(T)_\alpha{}^\beta = (L^{ab}_gT_aT_b)_\alpha{}^\beta \quad \longrightarrow \quad D_g(T)_{\alpha I}{}^{\beta J} \\
D_g(T)_{\alpha I}{}^{\beta J} = \left(L^{aM,bN}_g T_{aM}T_{bN}\right)_{\alpha I}{}^{\beta J} = \frac{(\eta^{ab}_IT^I_aT^I_b)_{\alpha(I)}{}^{\beta(I)}}{2k_I + Q_I}\delta^J_I = \Delta_{\mathfrak{g}^I}(T^I)\delta_{\alpha(I)}{}^{\beta(I)}\delta^J_I\,,
\end{gathered} \tag{2.8}$$

where $: \cdot :$ is operator product normal ordering and L^{ab}_g is the inverse inertia tensor of the affine-Sugawara constructions. The quantity $D_g(T)$ is the *conformal weight*

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matrix corresponding to representation T of g and $\Delta_{\mathfrak{g}^I}(T^I)$ is the conformal weight of representation T^I under $\mathfrak{g}^I$.

For permutation invariant systems, these relations take the simple form

$$\mathfrak{g}^I \simeq \mathfrak{g}\,, \quad a(I) = a\,, \quad \alpha(I) = \alpha \tag{2.9}$$

$$\eta^I_{ab} = \eta_{ab}\,, \quad \eta_I^{ab} = \eta^{ab}\,, \quad f^{I\ c}_{ab} = f_{ab}{}^c\,, \quad T^I_a = T_a\,, \quad k_I = k \tag{2.10}$$

$$[T_a, T_b] = if_{ab}{}^c T_c\,, \quad \mathrm{Tr}(T_a T_b) = y(T)\eta_{ab}\,, \quad M(k,T) = \frac{k}{y(T)}\mathbb{1} \tag{2.11}$$

$$L_g^{aI,bJ} = \frac{\eta^{ab}\delta^{IJ}}{2k + Q_{\mathfrak{g}}}, \qquad D_g(T)_{\alpha I}{}^{\beta J} = \frac{(\eta^{ab}T_a T_b)_\alpha{}^\beta}{2k + Q_{\mathfrak{g}}}\delta_I^J = \Delta_{\mathfrak{g}}(T)\delta_\alpha^\beta \delta_I^J\,, \tag{2.12}$$

where $\Delta_{\mathfrak{g}}(T)$ is the conformal weight of irrep T under any copy $\mathfrak{g}^I \simeq \mathfrak{g}$. In this case, the data matrix and the conformal weight matrix are proportional to 1, and the last relations in eqs. (2.11) and (2.12) hold for simple g as well.

2.2 WZW operator algebra

For general semisimple g, we have the OPEs

$$J_a(z)J_b(w) = \frac{G_{ab}}{(z-w)^2} + \frac{if_{ab}{}^c J_c(w)}{z-w} + \mathcal{O}(z-w)^0 \tag{2.13a}$$

$$\bar{J}_a(\bar{z})\bar{J}_b(\bar{w}) = \frac{G_{ab}}{(\bar{z}-\bar{w})^2} + \frac{if_{ab}{}^c \bar{J}_c(\bar{w})}{\bar{z}-\bar{w}} + \mathcal{O}(\bar{z}-\bar{w})^0, \qquad J_a(z)\bar{J}_b(\bar{z}) = \text{regular} \tag{2.13b}$$

$$J_a(z)g(T,\bar{w},w) = \frac{g(T,\bar{w},w)}{z-w}T_a + \mathcal{O}(z-w)^0\,,$$

$$\bar{J}_a(\bar{z})g(T,\bar{w},w) = -T_a\frac{g(T,\bar{w},w)}{\bar{z}-\bar{w}} + \mathcal{O}(\bar{z}-\bar{w})^0 \tag{2.14a}$$

$$T(z)g(T,\bar{w},w) = \frac{g(T,\bar{w},w)D_g(T)}{(z-w)^2} + \frac{\partial_w}{z-w}g(T,\bar{w},w) + \mathcal{O}(z-w)^0 \tag{2.14b}$$

$$\bar{T}(\bar{z})g(T,\bar{w},w) = \left(\frac{D_g(T)}{(\bar{z}-\bar{w})^2} + \frac{\partial_{\bar{w}}}{\bar{z}-\bar{w}}\right)g(T,\bar{w},w) + \mathcal{O}(\bar{z}-\bar{w})^0\,, \tag{2.14c}$$

where $J(z)$, $\bar{J}(\bar{z})$ are the currents of affine $(g \oplus g)$ and $g(T,\bar{z},z)$ is the affine primary field corresponding to matrix representation T. The currents are also Virasoro primary with conformal weight one under T and $\bar{T}$. As implied by eq. (2.14a), the affine primary field is also a direct sum

$$g(T,\bar{z},z)_\alpha{}^\beta \longrightarrow g(T,\bar{z},z)_{\alpha I}{}^{\beta J} = g_I(T^I,\bar{z},z)_{\alpha(I)}{}^{\beta(I)}\delta_I{}^J\,, \quad [M(k,T), g(T,\bar{z},z)] = 0 \tag{2.15}$$

of the affine primary fields $g_I(T^I,\bar{z},z)$ of $\mathfrak{g}^I$. In what follows, we often refer to the affine primary fields $g(T,\bar{z},z)$ as the WZW vertex operators.

The OPEs eq. (2.13) and eq. (2.14) are equivalent to many familiar commutation relations, among which we list for reference

$$J_a(z) = \sum_{m\in\mathbb{Z}} J_a(m) z^{-m-1}, \qquad \bar{J}_a(z) = \sum_{m\in\mathbb{Z}} \bar{J}_a(m) \bar{z}^{-m-1} \tag{2.16a}$$

$$T(z) = \sum_{m\in\mathbb{Z}} L(m) z^{-m-2}, \qquad \bar{T}(z) = \sum_{m\in\mathbb{Z}} \bar{L}(m) z^{-m-2} \tag{2.16b}$$

$$[J_a(m), J_b(n)] = i f_{ab}{}^c J_c(m+n) + m G_{ab} \delta_{m+n,0} \tag{2.16c}$$

$$[\bar{J}_a(m), \bar{J}_b(n)] = i f_{ab}{}^c \bar{J}_c(m+n) + m G_{ab} \delta_{m+n,0}, \qquad [J_a(m), \bar{J}_b(n)] = 0 \tag{2.16d}$$

$$\partial g(T,\bar{z},z) = [L(-1), g(T,\bar{z},z)], \qquad \bar{\partial} g(T,\bar{z},z) = [\bar{L}(-1), g(T,\bar{z},z)] \tag{2.16e}$$

$$[J_a(m), g(T,\bar{z},z)] = g(T,\bar{z},z)\ T_a z^m, \qquad [\bar{J}_a(m), g(T,\bar{z},z)] = -\bar{z}^m T_a\ g(T,\bar{z},z)\,, \tag{2.16f}$$

where eq. (2.16c) and eq. (2.16d) is the algebra of affine $(g \oplus g)$. Then (see for example ref. [36]) we may use (2.16e) and (2.16f) to obtain the WZW vertex operator equations

$$\partial g(T,\bar{z},z) = 2L_g^{ab} : J_a\ g(T,\bar{z},z) T_b :, \qquad \bar{\partial} g(T,\bar{z},z) = -2L_g^{ab} : \bar{J}_a T_b\ g(T,\bar{z},z) : \tag{2.17}$$

from which the Knizhnik-Zamolodchikov (KZ) equations [10] for WZW correlators can be derived [36].

3. The automorphism group H

3.1 Action in representation T

Let $H \subset \mathrm{Aut}(g)$ be a symmetry group (automorphism group) of the WZW model $A_g(H)$ and pick one representative $h_\sigma \in H$ of each conjugacy class of H. The actions of the H symmetry on the currents J, $\bar{J}$ and the affine primary fields $g(T)$ are

$$J_a(z)' = w(h_\sigma)_a{}^b J_b(z), \qquad \bar{J}_a(\bar{z})' = w(h_\sigma)_a{}^b \bar{J}_b(\bar{z}), \qquad w^\dagger(h_\sigma) w(h_\sigma) = 1 \tag{3.1a}$$

$$g(T,\bar{z},z)' = W(h_\sigma;T) g(T,\bar{z},z) W^\dagger(h_\sigma;T), \qquad W^\dagger(h_\sigma;T) W(h_\sigma;T) = 1\!\!1 \tag{3.1b}$$

$$w(h_\sigma), W(h_\sigma;T) \in H \subset \mathrm{Aut}(g), \quad \sigma = 0,\ldots,N_c - 1\,, \tag{3.1c}$$

where N_c is the number of conjugacy classes of H. The matrix $w(h_\sigma)$ is the action of $h_\sigma \in H$ in the adjoint of g, while $W(h_\sigma;T)$ is the action[1] of $h_\sigma \in H$ in matrix

[1]The explicit form of $W(T)$ is given for permutation subgroups of $\mathrm{Aut}(g)$ in subsection 7.1 and for inner-automorphic subgroups of $\mathrm{Aut}(g)$ in subsection 10.1. To use the formalism for automorphism groups which include outer automorphisms of simple $\mathfrak{g}$, one often needs to include reducible representations of $\mathfrak{g}$. As an example, we mention the charge conjugation automorphism $T^{(0)} \leftrightarrow \bar{T}^{(0)} = -T^{(0)\mathrm{T}}$ for the 3 and $\bar{3}$ of $SU(3)$ in the standard cartesian basis. In this case we

representation T of g. Automorphic invariance of the current-current OPEs (2.13a) then requires

$$w(h_\sigma)_a{}^c w(h_\sigma)_b{}^d G_{cd} = G_{ab}\,, \quad w(h_\sigma)_a{}^d w(h_\sigma)_b{}^e f_{de}{}^f w^\dagger(h_\sigma)_f{}^c = f_{ab}{}^c\,,$$
$$\forall\ h_\sigma \in H \subset \mathrm{Aut}(g)\,. \tag{3.2}$$

In what follows, these relations are called the H symmetry of G and f. The invariance of G_{ab} also guarantees the H invariance [13] of the inverse inertia tensor and the stress tensors:

$$L_g^{cd} w(h_\sigma)_c{}^a w(h_\sigma)_d{}^b = L_g^{ab}\,, \quad T(z)' = T(z)\,, \quad \bar{T}(\bar{z})' = \bar{T}(\bar{z})\,. \tag{3.3}$$

This much is well understood in current-algebraic orbifold theory [31, 32].

Here we extend this discussion to include the automorphic invariance of the Jg and $\bar{J}g$ OPEs, e.g.

$$J_a(z)' g(T,\bar{w},w)' = \frac{g(T,\bar{w},w)'}{z-w} T_a + \mathcal{O}(z-w)^0\,. \tag{3.4}$$

The invariance of these OPEs requires the *linkage relation*

$$W^\dagger(h_\sigma;T) T_a W(h_\sigma;T) = w(h_\sigma)_a{}^b T_b\,, \quad \forall\ h_\sigma \in H \subset \mathrm{Aut}(g) \tag{3.5a}$$

$$w^\dagger(h_\sigma)_a{}^b (W^\dagger(h_\sigma;T) T_b W(h_\sigma;T)) = w(h_\sigma)_a{}^b (W(h_\sigma;T) T_b W^\dagger(h_\sigma;T)) = T_a \tag{3.5b}$$

which constrains the action W given the action w. Here (3.5a), (3.5b) are equivalent forms of (3.5a), and we will refer to the equivalent forms in (3.5b) as the H-invariance of the representation matrices T.

The linkage relation tells us that W reduces to w

$$W(h_\sigma;T^{\mathrm{adj}}) = w(h_\sigma)\,, \quad (T_a^{\mathrm{adj}})_b{}^c \equiv -i f_{ab}{}^c \tag{3.6}$$

when T is taken to be the adjoint representation. In this case, the linkage relation is equivalent to the H-symmetry of the structure constants f in (3.2). One also finds that

$$T^{\mathrm{adj}}(s) \equiv s T^{\mathrm{adj}} s^{-1} \longrightarrow W(h_\sigma;T^{\mathrm{adj}}(s)) = s w(h_\sigma) s^{-1} \tag{3.7}$$

for similarity-transformed versions of the adjoint representation.

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may take

$$J_a' = \omega_a{}^b J_b, \qquad T_a = \begin{pmatrix} T_a^{(0)} & 0 \\ 0 & \bar{T}_a^{(0)} \end{pmatrix}, \qquad a = 1 \ldots \dim \mathfrak{g}$$
$$W(T) = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad W^\dagger(T) T_a W(T) = \omega_a{}^b T_b = \begin{pmatrix} \bar{T}_a^{(0)} & 0 \\ 0 & T_a^{(0)} \end{pmatrix}$$
$$\omega_A{}^A = -\omega_I{}^I = 1, \qquad \bar{T}_A = T_A, \qquad \bar{T}_I = -T_I, \qquad A = 2,5,7, \qquad I = 1,3,4,6,8$$

as the realization of $W(T)$ and the linkage relation (3.5a). We intend to return to this and related examples elsewhere.

The linkage relation also gives us information about the orders $\rho(\sigma)$ and $R(T,\sigma)$ of $w(h_\sigma)$ and $W(h_\sigma;T)$ respectively. These are the smallest integers which satisfy

$$w(h_\sigma)^{\rho(\sigma)} = 1, \quad W(h_\sigma;T)^{R(T,\sigma)} = \mathbb{1}, \quad R(T^{\text{adj}},\sigma) = \rho(\sigma)\,. \tag{3.8}$$

Using these definitions and the linkage relation, we find in general that

$$[W(h_\sigma;T)^{\rho(\sigma)}, T_a] = 0, \quad w(h_\sigma)^{R(T,\sigma)} = 1\,, \quad R(T,\sigma)/\rho(\sigma) \in \mathbb{Z}_{>0} \tag{3.9}$$

so that $R(T,\sigma)$ can be a positive multiple of $\rho(\sigma)$. In what follows, we will often abbreviate $R(\sigma) \equiv R(T,\sigma)$.

Taken together, the linkage relation, the normalization condition in (2.2) and the H-invariance (3.2) of G_{ab} tell us that the data matrix M

$$W^\dagger(h_\sigma;T)M(k,T)W(h_\sigma;T) = M(k,T)\,, \quad \forall\; h_\sigma \in H \subset \text{Aut}(g) \tag{3.10}$$

is also invariant under transformations in H.

Finally, automorphic invariance of the Tg and $\bar{T}g$ OPEs tells us that the conformal weight matrix $D_g(T)$ is also invariant

$$W(h_\sigma;T)D_g(T)W^\dagger(h_\sigma;T) = D_g(T)\,. \tag{3.11}$$

According to (2.8), this invariance also follows from the linkage relation and the invariance (3.3) of the inverse inertia tensor.

3.2 The extended H-eigenvalue problem

For the action $w(h_\sigma)$ of $h_\sigma \in H$ in the adjoint, the *H-eigenvalue problem*

$$w(h_\sigma)_a{}^b U^\dagger(\sigma)_b{}^{n(r)\mu} = U^\dagger(\sigma)_a{}^{n(r)\mu} E_{n(r)}(\sigma)\,, \quad \sigma = 0,\ldots,N_c-1 \tag{3.12a}$$

$$E_{n(r)}(\sigma) = e^{-2\pi i \frac{n(r)}{\rho(\sigma)}}\,, \quad E_{n(r)\pm\rho(\sigma)}(\sigma) = E_{n(r)}(\sigma)\,,$$

$$U^\dagger(\sigma)_a{}^{n(r)\pm\rho(\sigma),\mu} = U^\dagger(\sigma)_a{}^{n(r)\mu} \tag{3.12b}$$

$$U^\dagger(\sigma)U(\sigma) = 1\,, \quad U^\dagger(0) = 1\,, \quad E_{n(r)}(0) = \rho(0) = 1 \tag{3.12c}$$

$$U(\sigma)_{n(r)\mu}{}^a U^\dagger(\sigma)_a{}^{n(s)\nu} = \delta_{n(r)\mu}{}^{n(s)\nu}\,, \quad \delta_{n(r)\mu}{}^{n(s)\nu} \equiv \delta_{n(r)-n(s)\,,0\,\text{mod}\,\rho(\sigma)}\delta_\mu{}^\nu \tag{3.12d}$$

$$\bar{n}(r) = n(r) - \rho(\sigma)\lfloor \tfrac{n(r)}{\rho(\sigma)} \rfloor, \quad \bar{n}(r) \in \{0,\ldots,\rho(\sigma)-1\} \tag{3.12e}$$

was defined and studied in refs. [31], [35] and [32]. Here $r \equiv r(\sigma)$ runs over the degenerate subspaces of the H-eigenvalue problem and $n(r) \equiv n(r(\sigma)) \in \mathbb{Z}$. The unitary eigenvector matrix $U^\dagger(\sigma)$ and the eigenvalues $E(\sigma)$ are periodic with period $\rho(\sigma)$ in the spectral index $n(r)$, while $\mu = \mu(r(\sigma))$ is a degeneracy index. The pullbacks $\bar{n}(r)$ are identified in ref. [32] as the twist classes of the twisted currents of sector σ, where $\sigma = 0$ is the untwisted sector.

Correspondingly, we define now the *extended H-eigenvalue problem*

$$W(h_\sigma;T)_\alpha{}^\beta U^\dagger(T,\sigma)_\beta{}^{N(r)\mu} = U^\dagger(T,\sigma)_\alpha{}^{N(r)\mu} E_{N(r)}(T,\sigma)\,, \quad \sigma = 0,\ldots,N_c-1$$

$$ \tag{3.13a} $$

$$E_{N(r)}(T,\sigma) = e^{-2\pi i \frac{N(r)}{R(\sigma)}}\,, \quad N(r) \equiv N(T,r(\sigma)) \in \mathbb{Z} \tag{3.13b}$$

$$E_{N(r)\pm R(\sigma)}(T,\sigma) = E_{N(r)}(T,\sigma)\,, \quad U^\dagger(T,\sigma)_\alpha{}^{N(r)\pm R(\sigma),\mu} = U^\dagger(T,\sigma)_\alpha{}^{N(r)\mu} \tag{3.13c}$$

$$U^\dagger(T,\sigma)U(T,\sigma) = 1\!\!1\,, \quad U^\dagger(T,0) = 1\!\!1\,, \quad E_{N(r)}(T,0) = R(T,0) = 1 \tag{3.13d}$$

$$U(T,\sigma)_{N(r)\mu}{}^\alpha U^\dagger(T,\sigma)_\alpha{}^{N(s)\nu} = \delta_{N(r)\mu}{}^{N(s)\nu}\,,\ \delta_{N(r)\mu}{}^{N(s)\nu} \equiv \delta_{N(r)-N(s),0 \bmod R(\sigma)}\delta_\mu{}^\nu \tag{3.13e}$$

$$\bar{N}(r) = N(r) - R(\sigma)\lfloor \tfrac{N(r)}{R(\sigma)} \rfloor, \quad \bar{N}(r) \in \{0,\ldots,R(\sigma)-1\} \tag{3.13f}$$

$$\begin{aligned} WU^\dagger &= U^\dagger E\,,\ UW = EU,\ U^*W^* = E^*U^*\,,\ UW^\dagger = E^*U\,, \\ W^\dagger U^\dagger &= U^\dagger E^* \end{aligned} \tag{3.13g}$$

$$E(T,\sigma)_{N(r)\mu}{}^{N(s)\nu} \equiv \delta_{N(r)\mu}{}^{N(s)\nu} E_{N(r)}(T,\sigma) \tag{3.13h}$$

for the action $W(h_\sigma;T)$ of $h_\sigma \in H$ in matrix representation T. Here, the order $R(\sigma) \equiv R(T,\sigma)$, the spectral index $N(r) \equiv N(T,r(\sigma))$ and the degeneracy index $\mu \equiv \mu(T,r(\sigma))$ can all depend on T. The relations in (3.13g) are useful alternate forms of the extended eigenvalue problem.

Because of the linkage relation (3.5), the two eigenvalue problems (3.12) and (3.13) are not independent. In particular, the extended problem includes the original problem as the special case when $T = T^{\mathrm{adj}}$:

$$\begin{aligned} W(h_\sigma;T^{\mathrm{adj}}) &= w(h_\sigma)\,, & R(T^{\mathrm{adj}},\sigma) &= \rho(\sigma)\,, & (T_a^{\mathrm{adj}})_b{}^c &= -if_{ab}{}^c \\ E(T^{\mathrm{adj}},\sigma) &= E(\sigma)\,, & N(r) &= n(r)\,, & U^\dagger(T^{\mathrm{adj}},\sigma) &= U^\dagger(\sigma)\,. \end{aligned} \tag{3.14}$$

Solutions of the extended H-eigenvalue problem are discussed for general permutation groups and general groups of inner automorphisms in sections 7 and 10 respectively.

It is known [31, 32] that the eigenvectors $U^\dagger(\sigma)$ of the H-eigenvalue problem are Fourier basis elements with periodicity $\rho(\sigma)$ in the spectral index $n(r)$, and similarly the eigenvectors $U^\dagger(T,\sigma)$ of the extended problem are Fourier basis elements with periodicity $R(\sigma) = R(T,\sigma)$ in the spectral index $N(r)$. In what follows, all quantities inherit the periodicities

$$N(r) \longrightarrow N(r) \pm R(\sigma)\,, \quad n(r) \longrightarrow n(r) \pm \rho(\sigma) \tag{3.15}$$

in the spectral indices $\{n(r)\}$, $\{N(r)\}$ of the H-eigenvalue problems. Since $R(T,\sigma)$ can be a multiple of $\rho(\sigma)$, this means in particular that all the orbifold duality transformations of sector σ are discrete Fourier transforms with fundamental period $\rho(\sigma)$.

We will also need the behavior of the eigenvalue problems under conjugation in H:

$$w(h_\sigma) \longrightarrow v^\dagger(\sigma)w(h_\sigma)v(\sigma)\,, \quad v^\dagger(\sigma)v(\sigma) = 1\,, \quad v(\sigma) \in H \tag{3.16a}$$

$$U^\dagger(\sigma) \longrightarrow v^\dagger(\sigma)U^\dagger(\sigma)\,, \quad U(\sigma) \longrightarrow U(\sigma)v(\sigma) \tag{3.16b}$$

$$v(\sigma)_a{}^c v(\sigma)_b{}^d G_{cd} \;=\; G_{ab}\,, \quad v(\sigma)_a{}^d v(\sigma)_b{}^e f_{de}{}^f v^\dagger(\sigma)_f{}^c = f_{ab}{}^c \tag{3.16c}$$

$$W(h_\sigma;T) \longrightarrow v^\dagger(T,\sigma)W(h_\sigma;T)v(T,\sigma)\,, \quad v^\dagger(T,\sigma)v(T,\sigma) = 1\!\!1\,, \quad v(T,\sigma) \in H \tag{3.16d}$$

$$U^\dagger(T,\sigma) \longrightarrow v^\dagger(T,\sigma)U^\dagger(T,\sigma)\,, \quad U(T,\sigma) \longrightarrow U(T,\sigma)v(T,\sigma) \tag{3.16e}$$

$$v^\dagger(T,\sigma)T_a v(T,\sigma) \;=\; v(\sigma)_a{}^b T_b\,. \tag{3.16f}$$

The relations in (3.16c) follow from (3.2) because $v(\sigma) \in H$, and the first three relations in (3.16) were given in Refs. [31] and [32]. Taken together, the conjugation relations (3.16) preserve the linkage relation (3.5), and will allow us to check that all the orbifold duality transformations of this paper are class functions.

4. Eigenfields

4.1 Eigencurrents and eigenprimary fields

Following refs. [31] and [32], we define the eigencurrents $\mathcal{J}(\sigma)$, $\bar{\mathcal{J}}(\sigma)$ and eigenprimary fields $\mathcal{G}(T,\sigma)$ of sector σ as follows

$$\begin{aligned} \mathcal{J}_{n(r)\mu}(z,\sigma) &\equiv \chi(\sigma)_{n(r)\mu} U(\sigma)_{n(r)\mu}{}^a J_a(z)\,, \\ \bar{\mathcal{J}}_{n(r)\mu}(\bar z,\sigma) &\equiv \chi(\sigma)_{n(r)\mu} U(\sigma)_{n(r)\mu}{}^a \bar J_a(\bar z) \end{aligned} \tag{4.1}$$

$$\mathcal{G}(T,\bar z,z,\sigma) \equiv U(T,\sigma)g(T,\bar z,z)U^\dagger(T,\sigma) \tag{4.2}$$

$$\mathcal{G}(T,\bar z,z,\sigma)_{N(r)\mu}{}^{N(s)\nu} = U(T,\sigma)_{N(r)\mu}{}^\alpha g(T,\bar z,z,\sigma)_\alpha{}^\beta U^\dagger(T,\sigma)_\beta{}^{N(s)\nu} \tag{4.3}$$

$$\chi(\sigma)_{n(r)\pm\rho(\sigma),\mu} = \chi(\sigma)_{n(r)\mu}\,; \quad \chi(0)_{n(r)\mu} = 1\,, \tag{4.4}$$

where $U^\dagger$ and $U^\dagger(T)$ are the eigenvector matrices respectively of the H-eigenvalue problem (3.12) and its extension (3.13). The quantities $\chi(\sigma)$,which satisfy the conventions in (4.4), are otherwise arbitrary normalization constants.

The responses of the eigenfields to elements of the automorphism group are

$$\begin{aligned} \mathcal{J}_{n(r)\mu}(z,\sigma)' &= E_{n(r)}(\sigma)\mathcal{J}_{n(r)\mu}(z,\sigma) = e^{-2\pi i\frac{n(r)}{\rho(\sigma)}}\mathcal{J}_{n(r)\mu}(z,\sigma) \\ \bar{\mathcal{J}}_{n(r)\mu}(\bar z,\sigma)' &= E_{n(r)}(\sigma)\bar{\mathcal{J}}_{n(r)\mu}(\bar z,\sigma) = e^{-2\pi i\frac{n(r)}{\rho(\sigma)}}\bar{\mathcal{J}}_{n(r)\mu}(z,\sigma) \\ \mathcal{G}(T,\bar z,z,\sigma)' &= E(T,\sigma)\mathcal{G}(T,\bar z,z,\sigma)E(T,\sigma)^* \\ (\mathcal{G}(T,\bar z,z,\sigma)')_{N(r)\mu}{}^{N(s)\nu} &= e^{-\frac{2\pi i}{R(\sigma)}(N(r)-N(s))}\mathcal{G}(T,\bar z,z,\sigma)_{N(r)\mu}{}^{N(s)\nu}\,. \end{aligned} \tag{4.5}$$

The diagonal response of the eigencurrents is well known in the theory of current-algebraic orbifolds [31, 32], while the two-sided response of the eigenprimary fields is new.

The next stage in the orbifold program is to rewrite all relations in the symmetric theory in terms of the corresponding eigenfields. This step gives rise to the *orbifold* *duality transformations* [31, 32], which express the twisted tensors in the orbifold as discrete Fourier transforms of corresponding H-symmetric tensors in the symmetric theory. Moreover, the twisted tensors satisfy *selection rules* [31, 32] which are dual to the H-symmetry of the original tensors.

The case of the eigencurrent OPEs was worked out in refs. [31] and [32]. Here one encounters the orbifold duality transformations from the metric G and structure constants f to the *twisted metric* $\mathcal{G}(\sigma)$ and the *twisted structure constants* $\mathcal{F}(\sigma)$ of sector σ

$$G \underset{\sigma}{\longrightarrow} \mathcal{G}(G,\sigma)\,, \quad f \underset{\sigma}{\longrightarrow} \mathcal{F}(f,\sigma)\,. \tag{4.6}$$

The explicit form of these duality transformations is [31, 32]

$$\begin{aligned}
\mathcal{G}_{n(r)\mu;n(s)\nu}(\sigma) &= \chi(\sigma)_{n(r)\mu}\chi(\sigma)_{n(s)\nu}U(\sigma)_{n(r)\mu}{}^{a}U(\sigma)_{n(s)\nu}{}^{b}G_{ab} &\text{(4.7a)}\\
&= \delta_{n(r)+n(s),0 \bmod \rho(\sigma)}\mathcal{G}_{n(r)\mu;-n(r),\nu}(\sigma) &\text{(4.7b)}\\
\mathcal{G}^{n(r)\mu;n(s)\nu}(\sigma) &= \chi(\sigma)^{-1}_{n(r)\mu}\chi(\sigma)^{-1}_{n(s)\nu}G^{ab}U^{\dagger}(\sigma)_{a}{}^{n(r)\mu}U^{\dagger}(\sigma)_{b}{}^{n(s)\nu} &\text{(4.7c)}\\
&= \delta_{n(r)+n(s),0 \bmod \rho(\sigma)}\mathcal{G}^{n(r)\mu;-n(r),\nu}(\sigma) &\text{(4.7d)}\\
\mathcal{G}^{n(r)\mu;n(t)\delta}(\sigma)\mathcal{G}_{n(t)\delta;n(s)\nu}(\sigma) &= \delta_{n(s)\nu}{}^{n(r)\mu} &\text{(4.7e)}\\
\mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(t)\delta}(\sigma) &= \chi(\sigma)_{n(r)\mu}\chi(\sigma)_{n(s)\nu}\chi(\sigma)^{-1}_{n(t)\delta}\times\\
&\quad\times U(\sigma)_{n(r)\mu}{}^{a}U(\sigma)_{n(s)\nu}{}^{b}f_{ab}{}^{c}U^{\dagger}(\sigma)_{c}{}^{n(t)\delta} &\text{(4.7f)}\\
&= \delta_{n(r)+n(s)-n(t),0 \bmod \rho(\sigma)}\mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(r)+n(s),\delta}(\sigma)\,. &\text{(4.7g)}
\end{aligned}$$

The forms (4.7b), (4.7d) and (4.7g) are solutions to selection rules which are dual to the H-invariance of G and f in (3.2). Twisted tensor indices can be lowered and raised with the twisted metric and its inverse, and the twisted structure constants with all indices down are totally antisymmetric. The twisted metric and twisted structure constants are also class functions [31, 32] under conjugation in H

$$\mathcal{G}(U(\sigma)v(\sigma);\sigma) = \mathcal{G}(U(\sigma);\sigma)\,, \qquad \mathcal{F}(U(\sigma)v(\sigma);\sigma) = \mathcal{F}(U(\sigma);\sigma) \tag{4.8}$$

according to eqs. (3.2), (3.16) and (4.7).

4.2 The twisted representation matrices of sector σ

We consider next the *twisted representation matrices* $\mathcal{T}$, which are duality transformations

$$T \underset{\sigma}{\longrightarrow} \mathcal{T}(T,\sigma) \tag{4.9}$$

from the untwisted representation matrices T. The explicit form of the twisted representation matrices is

$$\mathcal{T}_{n(r)\mu}(T,\sigma) \equiv \chi(\sigma)_{n(r)\mu}U(\sigma)_{n(r)\mu}{}^{a}\Big(\,U(T,\sigma)T_{a}U^{\dagger}(T,\sigma)\,\Big)\,. \tag{4.10}$$

These objects appear for example in the OPEs of the left mover eigencurrents with the eigenprimary fields

$$
\begin{aligned}
\mathcal{G}(\mathcal{T},\bar{z},z,\sigma) &\equiv \mathcal{G}(T(\mathcal{T}),\bar{z},z,\sigma) \\
\mathcal{J}_{n(r)\mu}(z,\sigma)\mathcal{G}(\mathcal{T},\bar{w},w,\sigma) &= \frac{\mathcal{G}(\mathcal{T},\bar{w},w,\sigma)}{z-w}\mathcal{T}_{n(r)\mu}(T,\sigma)+\mathcal{O}(z-w)^0 \qquad (4.11)
\end{aligned}
$$

and will play a central role in the orbifold dynamics below. Examples of $\mathcal{T}$ for specific orbifolds are found in sections 7 and 10.

The twisted representation matrices satisfy many important relations, including

$$
\left[\mathcal{T}_{n(r)\mu}(T,\sigma),\mathcal{T}_{n(s)\nu}(T,\sigma)\right] = i\mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(r)+n(s),\delta}(\sigma)\mathcal{T}_{n(r)+n(s),\delta}(T,\sigma) \qquad (4.12)
$$

$$
[\mathcal{T}_{n(r)\mu}(T,\sigma),\mathcal{T}_{n(s)\nu}(T,\sigma)\mathcal{G}^{n(s)\nu;n(t)\delta}(\sigma)\mathcal{T}_{n(t)\delta}(T,\sigma)] = 0\,, \qquad (4.13)
$$

where $\mathcal{G}(\sigma)$ and $\mathcal{F}(\sigma)$ are the twisted metric and twisted structure constants of sector σ in (4.7).

The normalization condition for the twisted representation matrices

$$
\begin{aligned}
\widehat{\mathrm{Tr}}(\mathcal{M}(\mathcal{T},\sigma)\mathcal{T}_{n(r)\mu}(T,\sigma)\mathcal{T}_{n(s)\nu}(T,\sigma)) &= \mathcal{G}_{n(r)\mu;n(s)\nu}(\sigma) \\
&= \delta_{n(r)+n(s),0 \bmod \rho(\sigma)}\,\mathcal{G}_{n(r)\mu;-n(r),\nu}(\sigma) \qquad (4.14) \\
\widehat{\mathrm{Tr}}(AB) &\equiv \sum_{r,\mu,s,\nu} A_{N(r)\mu}{}^{N(s)\nu}B_{N(s)\nu}{}^{N(r)\mu}\,, \\
[\mathcal{M}(\mathcal{T},\sigma),\mathcal{T}_{n(r)\mu}(T,\sigma)] &= 0 \qquad (4.15)
\end{aligned}
$$

follows from (2.2), (4.10) and (4.7a). Here we have encountered the *twisted data matrix* $\mathcal{M}(\mathcal{T},\sigma)$

$$
M \underset{\sigma}{\longrightarrow} \mathcal{M}(M,\sigma)\;:\mathcal{M}(\mathcal{T},\sigma) \equiv \mathcal{M}(T(\mathcal{T}),\sigma) \equiv U(T,\sigma)M(k,T)U^{\dagger}(T,\sigma) \qquad (4.16)
$$

$$
\mathcal{M}(\mathcal{T},\sigma)_{N(r)\mu}{}^{N(s)\nu} = \sum_I \frac{k_I}{y_I(T^I)}\sum_\alpha U(T,\sigma)_{N(r)\mu}{}^{\alpha I}U^{\dagger}(T,\sigma)_{\alpha I}{}^{N(s)\nu} \qquad (4.17)
$$

which is the duality transformation of the data matrix M in (2.1). Like M, the twisted data matrix is easily invertible. Other properties of the twisted data matrix include

$$
[\mathcal{M}(\mathcal{T},\sigma),\mathcal{G}(\mathcal{T},\bar{z},z,\sigma)] = 0, \qquad E(T,\sigma)\mathcal{M}(\mathcal{T},\sigma)E(T,\sigma)^* = \mathcal{M}(\mathcal{T},\sigma)\,. \qquad (4.18)
$$

The $\mathcal{M}$-selection rule in (4.18) is dual to the H-invariance (3.10) of the data matrix M. The twisted data matrix is also a class function

$$
\mathcal{M}(U(T,\sigma)v(T,\sigma);\sigma) = \mathcal{M}(U(T,\sigma);\sigma) \qquad (4.19)
$$

according to eqs. (3.10), (3.16) and (4.16).

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The twisted representation matrices also satisfy the selection rule

$$\mathcal{T}_{n(r)\mu}(T,\sigma) = E_{n(r)}(\sigma)\big(E(T,\sigma)\mathcal{T}_{n(r)\mu}(T,\sigma)E(T,\sigma)^* \big) \tag{4.20a}$$

$$\mathcal{T}_{n(r)\mu}(T,\sigma)_{N(s)\nu}{}^{N(t)\delta}(1 - e^{-2\pi i(\frac{n(r)}{\rho(\sigma)} + \frac{N(s)-N(t)}{R(\sigma)})}) = 0\,. \tag{4.20b}$$

This selection rule is dual to the linkage relation (3.5) and in particular to the forms (3.5b) of the H-invariance of the representation matrices T. To see this, insert the forms (3.5b) into the duality transformation (4.10) and use both eigenvalue problems (3.13) and (3.12).

The solution of the selection rule is the "Wigner-Eckart" relation

$$\mathcal{T}_{n(r)\mu}(T,\sigma)_{N(s)\nu}{}^{N(t)\delta} = \delta_{\frac{R(\sigma)}{\rho(\sigma)}n(r)+N(s)-N(t),0 \bmod R(\sigma)}\mathcal{T}_{n(r)\mu}(T,\sigma)_{N(s)\nu}{}^{N(s)+\frac{R(\sigma)}{\rho(\sigma)}n(r),\delta} \tag{4.21}$$

$$\delta_{\frac{R(\sigma)}{\rho(\sigma)}n(r)+N(s)-N(t),0 \bmod R(\sigma)} = \delta_{n(r)+\frac{\rho(\sigma)}{R(\sigma)}(N(s)-N(t)),0 \bmod \rho(\sigma)} = \delta_{\frac{n(r)}{\rho(\sigma)}+\frac{N(s)-N(t)}{R(\sigma)},0 \bmod 1}\,. \tag{4.22}$$

The twisted representation matrices are also class functions under conjugation in H

$$\mathcal{T}(U(\sigma)v(\sigma),U(T,\sigma)v(T,\sigma);\sigma) = \mathcal{T}(U(\sigma),U(T,\sigma);\sigma) \tag{4.23}$$

according to (3.16), (4.10) and the linkage relation (3.5). More generally, all the duality transformations of this paper are class functions, although we will not mention this explicitly below.

4.3 Other duality transformations

The affine-Sugawara stress tensors $T(z)$ and $\bar{T}(\bar{z})$ can easily be rewritten in terms of the eigencurrents, for example

$$\begin{aligned} T(z) &= \sum_{r,\mu,\nu} \mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)\mu;-n(r),\nu}(\sigma) : \mathcal{J}_{n(r)\mu}(z,\sigma)\mathcal{J}_{-n(r),\nu}(z,\sigma) : \\ &\equiv \mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)\mu;-n(r),\nu}(\sigma) : \mathcal{J}_{n(r)\mu}(z,\sigma)\mathcal{J}_{-n(r),\nu}(z,\sigma) : \end{aligned} \tag{4.24a}$$

$$\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)\mu;n(s)\nu}(\sigma) = \mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(s)\nu;n(r)\mu}(\sigma) = \chi(\sigma)_{n(r)\mu}^{-1}\chi(\sigma)_{n(s)\nu}^{-1}L_g^{ab}U^\dagger(\sigma)_a{}^{n(r)\mu}U^\dagger(\sigma)_b{}^{n(s)\nu} \tag{4.24b}$$

$$= \delta_{n(r)+n(s),0 \bmod \rho(\sigma)}\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)\mu;-n(r),\nu}(\sigma) \tag{4.24c}$$

where $\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}(\sigma)$ is the duality transformation called the *twisted inverse inertia tensor* [34, 31, 35, 32] of sector σ.

We also find the $T\mathcal{G}$ OPE

$$T(z)\mathcal{G}(\mathcal{T},\bar{w},w,\sigma)=\frac{\mathcal{G}(\mathcal{T},\bar{w},w,\sigma)\mathcal{D}_{\hat{\mathfrak{g}}(\sigma)}(\mathcal{T})}{(z-w)^2}+\frac{\partial_w}{z-w}\mathcal{G}(\mathcal{T},\bar{w},w,\sigma)+\mathcal{O}(z-w)^0 \tag{4.25}$$

Here we have encountered the *twisted conformal weight matrix* $\mathcal{D}_{\hat{\mathfrak{g}}(\sigma)}(\mathcal{T})$

$$D_g(T)\underset{\sigma}{\longrightarrow}\mathcal{D}_{\hat{\mathfrak{g}}(\sigma)}(\mathcal{T}) \tag{4.26a}$$

$$\mathcal{D}_{\hat{\mathfrak{g}}(\sigma)}(\mathcal{T})\equiv U(T,\sigma)D_g(T)U^\dagger(T,\sigma)=\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)\mu;-n(r),\nu}\mathcal{T}_{n(r)\mu}(T,\sigma)\mathcal{T}_{-n(r),\nu}(T,\sigma) \tag{4.26b}$$

which is the duality transformation of the conformal weight matrix in (2.8). The twisted conformal weight matrix satisfies the selection rule

$$\mathcal{D}_{\hat{\mathfrak{g}}(\sigma)}(\mathcal{T})_{N(r)\mu}{}^{N(s)\nu}=\delta_{N(r)-N(s),0\bmod R(\sigma)}\mathcal{D}_{\hat{\mathfrak{g}}(\sigma)}(\mathcal{T})_{N(r)\mu}{}^{N(r)\nu} \tag{4.27}$$

which is dual to the invariance (3.11) of the conformal weight matrix.

This completes the set of duality transformations that we will need to describe the WZW orbifolds in this paper. We finally note that all these twisted objects reduce to their original untwisted counterparts

$$\begin{array}{ccc}\mathcal{G}(0)=G & \mathcal{F}(0)=f & \mathcal{L}_{\hat{\mathfrak{g}}(0)}(0)=L_g\\ \mathcal{T}(T,0)=T & \mathcal{M}(T,0)=M(T) & \mathcal{D}_{\hat{\mathfrak{g}}(0)}(\mathcal{T})=D_g(T)\end{array} \tag{4.28}$$

in the untwisted sector $\sigma=0$ of each orbifold.

5. The general WZW orbifold $A_g(H)/H$

5.1 Local isomorphisms and monodromies

To go from the symmetric theory $A_g(H)$ in the eigenfield basis to sector σ of the WZW orbifold $A_g(H)/H$, we apply the principle of *local isomorphisms*[33, 31, 35, 32]. This principle tells us to replace all eigenfields by corresponding *twisted operators* with definite monodromy,

$$\begin{aligned}\mathcal{J}_{n(r)\mu}(z,\sigma),\bar{\mathcal{J}}_{n(r)\mu}(\bar{z},\sigma)&\underset{\sigma}{\longrightarrow}\hat{J}_{n(r)\mu}(z,\sigma),\hat{\bar{J}}_{n(r)\mu}(\bar{z},\sigma)\,,\\ \mathcal{G}(\mathcal{T},\bar{z},z,\sigma)&\underset{\sigma}{\longrightarrow}\hat{g}(\mathcal{T},\bar{z},z,\sigma)\,,\end{aligned} \tag{5.1}$$

where $\hat{J}(\sigma)$, $\hat{\bar{J}}(\sigma)$ are the twisted left and right mover currents and $\hat{g}(\mathcal{T},\bar{z},z,\sigma)$ is the *twisted affine primary field* corresponding to the twisted representation matrix $\mathcal{T}=\mathcal{T}(T,\sigma)$. By local isomorphisms from (4.18) the twisted affine primary fields satisfy

$$[\mathcal{M}(\mathcal{T},\sigma),\hat{g}(\mathcal{T},\bar{z},z,\sigma)]=0 \tag{5.2}$$

and all the twisted operators inherit the periodicities

$$\hat{J}_{n(r)\pm\rho(\sigma),\mu}(z,\sigma) = \hat{J}_{n(r)\mu}(z,\sigma)\,, \qquad \hat{\bar{J}}_{n(r)\pm\rho(\sigma),\mu}(\bar{z},\sigma) = \hat{\bar{J}}_{n(r)\mu}(\bar{z},\sigma) \tag{5.3}$$

$$\hat{g}(\mathcal{T},\bar{z},z,\sigma)_{N(r)\pm R(\sigma),\mu}{}^{N(s)\nu} = \hat{g}(\mathcal{T},\bar{z},z,\sigma)_{N(r)\mu}{}^{N(s)\pm R(\sigma),\nu} = \hat{g}(\mathcal{T},\bar{z},z,\sigma)_{N(r)\mu}{}^{N(s)\nu} \tag{5.4}$$

of the spectral indices $\{n(r)\}$ and $\{N(r)\}$.

For the twisted currents, the principle of local isomorphisms tells us to take the monodromies to be the old automorphic responses

$$\hat{J}_{n(r)\mu}(ze^{2\pi i},\sigma) = E_{n(r)}(\sigma)\hat{J}_{n(r)\mu}(z,\sigma) = e^{-2\pi i\frac{n(r)}{\rho(\sigma)}}\hat{J}_{n(r)\mu}(z,\sigma) \tag{5.5a}$$

$$\hat{\bar{J}}_{n(r)\mu}(\bar{z}e^{-2\pi i},\sigma) = E_{n(r)}(\sigma)\hat{\bar{J}}_{n(r)\mu}(\bar{z},\sigma) = e^{-2\pi i\frac{n(r)}{\rho(\sigma)}}\hat{\bar{J}}_{n(r)\mu}(\bar{z},\sigma)\,. \tag{5.5b}$$

The rule (5.5a) was verified for the twisted left mover currents in refs. [31, 32] and the rule (5.5b) says that the same monodromies are obtained for the twisted right mover currents when the same path is followed for z and $\bar{z} = z^*$. That this is the correct rule for the right mover currents can be argued both at the level of characters (see subsection 5.4) and at the classical level (see subsection 5.7) where it is understood as a consequence of locality.

Using (4.24) and the local isomorphisms, we obtain the orbifold stress tensors

$$T(z) \underset{\sigma}{\longrightarrow} \hat{T}_\sigma(z)\,, \qquad \bar{T}(\bar{z}) \underset{\sigma}{\longrightarrow} \hat{\bar{T}}_\sigma(\bar{z}) \tag{5.6}$$

$$\hat{T}_\sigma(z) = \mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)\mu;-n(r),\nu}(\sigma) : \hat{J}_{n(r)\mu}(z,\sigma)\hat{J}_{-n(r),\nu}(z,\sigma) :, \qquad \hat{T}_\sigma(ze^{2\pi i}) = \hat{T}_\sigma(z) \tag{5.7}$$

$$\hat{\bar{T}}_\sigma(\bar{z}) = \mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)\mu;-n(r),\nu}(\sigma) : \hat{\bar{J}}_{n(r)\mu}(\bar{z},\sigma)\hat{\bar{J}}_{-n(r),\nu}(\bar{z},\sigma) :, \qquad \hat{\bar{T}}_\sigma(\bar{z}e^{-2\pi i}) = \hat{\bar{T}}_\sigma(\bar{z}) \tag{5.8}$$

which are the *twisted affine-Sugawara constructions* [18, 24, 31, 32] of sector σ. Here, $\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}(\sigma)$ is the twisted inverse inertia tensor (4.24b) of the twisted affine-Sugawara construction and $: \cdot :$ is OPE normal ordering [34, 31, 35, 32] of the twisted currents.

The monodromies of the twisted primary fields $\hat{g}(\mathcal{T},\bar{z},z,\sigma)$ are quite complicated because the monodromies of the untwisted affine primary fields are complicated (see e.g. ref. [36]). The monodromies of $\hat{g}(\mathcal{T},\bar{z},z,\sigma)$ are however determined in principle by the twisted KZ equations discussed below. On the other hand, the monodromies of the classical limit of the affine primary fields (the WZW group elements) are trivial, so the *classical monodromies* of the classical limit of the twisted affine primary fields (the group orbifold elements) are easily deduced from the local isomorphisms (see subsection 5.7).

5.2 WZW orbifold OPEs

For sector σ of the general WZW orbifold $A_g(H)/H$, we then obtain the twisted current-current system

$$\hat{J}_{n(r)\mu}(z,\sigma)\hat{J}_{n(s)\nu}(w,\sigma) = \frac{\delta_{n(r)+n(s),0 \bmod \rho(\sigma)}\mathcal{G}_{n(r)\mu;-n(r),\nu}(\sigma)}{(z-w)^2} + \tag{5.9a}$$

$$+\frac{i\mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(r)+n(s),\delta}(\sigma)\hat{J}_{n(r)+n(s),\delta}(w,\sigma)}{z-w} + \mathcal{O}(z-w)^0$$

$$\hat{\bar{J}}_{n(r)\mu}(\bar{z},\sigma)\hat{\bar{J}}_{n(s)\nu}(\bar{w},\sigma) = \frac{\delta_{n(r)+n(s),0 \bmod \rho(\sigma)}\mathcal{G}_{n(r)\mu;-n(r),\nu}(\sigma)}{(\bar{z}-\bar{w})^2} + \tag{5.9b}$$

$$+\frac{i\mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(r)+n(s),\delta}(\sigma)\hat{\bar{J}}_{n(r)+n(s),\delta}(\bar{w},\sigma)}{\bar{z}-\bar{w}} + \mathcal{O}(\bar{z}-\bar{w})^0$$

$$\hat{J}_{n(r)\mu}(z,\sigma)\hat{\bar{J}}_{n(r)\mu}(\bar{w},\sigma) = \text{regular} \tag{5.9c}$$

by local isomorphisms from the corresponding OPEs of the eigencurrents. Here $\mathcal{G}(\sigma)$ and $\mathcal{F}(\sigma)$ are respectively the twisted metric and the twisted structure constants given in eq. (4.7) .

The twisted affine-Sugawara constructions $\hat{T}_\sigma$ and $\hat{\bar{T}}_\sigma$ satisfy the OPEs of Vir $\oplus$ Vir, as expected, with central charges

$$\hat{c}(\sigma) = \hat{\bar{c}}(\sigma) = 2\mathcal{G}_{n(r)\mu;-n(r),\nu}(\sigma)\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)\mu;-n(r),\nu}(\sigma) = 2G_{ab}L_g^{ab} = c_g\,, \quad \sigma = 0,\ldots,N_c-1 \tag{5.10}$$

which equal the untwisted affine-Sugawara central charge c_g. Moreover, we find the OPEs

$$\hat{T}_\sigma(z)\hat{J}_{n(r)\mu}(w,\sigma) = \left(\frac{1}{(z-w)^2} + \frac{\partial_w}{z-w}\right)\hat{J}_{n(r)\mu}(w,\sigma) + \mathcal{O}(z-w)^0 \tag{5.11a}$$

$$\hat{\bar{T}}_\sigma(\bar{z})\hat{\bar{J}}_{n(r)\mu}(\bar{w},\sigma) = \left(\frac{1}{(\bar{z}-\bar{w})^2} + \frac{\partial_{\bar{w}}}{\bar{z}-\bar{w}}\right)\hat{\bar{J}}_{n(r)\mu}(\bar{w},\sigma) + \mathcal{O}(\bar{z}-\bar{w})^0 \tag{5.11b}$$

$$\hat{T}_\sigma(z)\hat{\bar{J}}_{n(r)\mu}(\bar{w},\sigma) = \hat{\bar{T}}_\sigma(\bar{z})\hat{J}_{n(r)\mu}(w,\sigma) = \text{regular} \tag{5.11c}$$

of the twisted affine-Sugawara constructions with the twisted currents.

For the OPEs involving the twisted affine primary fields $\hat{g}(\mathcal{T},\bar{z},z,\sigma)$, we obtain

$$\hat{J}_{n(r)\mu}(z,\sigma)\hat{g}(\mathcal{T},\bar{w},w,\sigma) = \frac{\hat{g}(\mathcal{T},\bar{w},w,\sigma)}{z-w}\mathcal{T}_{n(r)\mu}(T,\sigma) + \mathcal{O}(z-w)^0 \tag{5.12a}$$

$$\hat{\bar{J}}_{n(r)\mu}(\bar{z},\sigma)\hat{g}(\mathcal{T},\bar{w},w,\sigma) = -\mathcal{T}_{n(r)\mu}(T,\sigma)\frac{\hat{g}(\mathcal{T},\bar{w},w,\sigma)}{\bar{z}-\bar{w}} + \mathcal{O}(\bar{z}-\bar{w})^0 \tag{5.12b}$$

$$\hat{T}_\sigma(z)\hat{g}(\mathcal{T},\bar{w},w,\sigma) = \frac{\hat{g}(\mathcal{T},\bar{w},w,\sigma)\mathcal{D}_{\hat{\mathfrak{g}}(\sigma)}(\mathcal{T})}{(z-w)^2} + \frac{\partial_w}{z-w}\hat{g}(\mathcal{T},\bar{w},w,\sigma) + \mathcal{O}(z-w)^0 \tag{5.12c}$$

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$$\hat{\bar{T}}_\sigma(\bar{z})\hat{g}(\mathcal{T},\bar{w},w,\sigma) = \left(\frac{\mathcal{D}_{\hat{\mathfrak{g}}(\sigma)}(\mathcal{T})}{(\bar{z}-\bar{w})^2} + \frac{\partial_{\bar{w}}}{\bar{z}-\bar{w}}\right)\hat{g}(\mathcal{T},\bar{w},w,\sigma) + \mathcal{O}(\bar{z}-\bar{w})^0 \,, \quad (5.12d)$$

where $\mathcal{T}(T,\sigma)$ and $\mathcal{D}_{\hat{\mathfrak{g}}(\sigma)}(\mathcal{T})$ are respectively the twisted representation matrices and twisted conformal weight matrices of sector σ.

The local isomorphisms also give the *twisted vertex operator equations*

$$\partial\hat{g}(\mathcal{T},\bar{z},z,\sigma) = 2\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)\mu;-n(r),\nu}(\sigma) : \hat{J}_{n(r)\mu}(z,\sigma)\hat{g}(\mathcal{T},\bar{z},z,\sigma)\mathcal{T}_{-n(r),\nu}(T,\sigma) : \quad (5.13a)$$

$$\bar{\partial}\hat{g}(\mathcal{T},\bar{z},z,\sigma) = -2\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)\mu;-n(r),\nu}(\sigma) : \hat{\bar{J}}_{n(r)\mu}(\bar{z},\sigma)\mathcal{T}_{-n(r),\nu}(T,\sigma)\hat{g}(\mathcal{T},\bar{z},z,\sigma) : (5.13b)$$

from the untwisted vertex operator equations (2.17). The normal ordering here is also OPE normal ordering [34, 31, 35, 32]. These world-sheet differential equations for the twisted affine primary fields are the WZW orbifold operator equations of motion, and they will be central in the derivation of the twisted KZ equations below.

5.3 Mode form of the general WZW orbifold algebra

Using the monodromies (5.5) and (5.7), (5.8) we find the mode relations:

$$\begin{aligned}
\hat{J}_{n(r)\mu}(z,\sigma) &= \sum_{m\in\mathbb{Z}} \hat{J}_{n(r)\mu}\left(m+\frac{n(r)}{\rho(\sigma)}\right) z^{-\left(m+\frac{n(r)}{\rho(\sigma)}\right)-1} \\
\hat{T}_\sigma(z) &= \sum_{m\in\mathbb{Z}} L_\sigma(m) z^{-m-2} &(5.14a)\\
\hat{\bar{J}}_{n(r)\mu}(\bar{z},\sigma) &= \sum_{m\in\mathbb{Z}} \hat{\bar{J}}_{n(r)\mu}\left(m+\frac{n(r)}{\rho(\sigma)}\right) \bar{z}^{\left(m+\frac{n(r)}{\rho(\sigma)}\right)-1} \\
\hat{\bar{T}}_\sigma(\bar{z}) &= \sum_{m\in\mathbb{Z}} \bar{L}_\sigma(m) \bar{z}^{-m-2} &(5.14b)
\end{aligned}$$

$$\hat{J}_{n(r)\pm\rho(\sigma),\mu}\left(m\mp 1+\frac{n(r)\pm\rho(\sigma)}{\rho(\sigma)}\right) = \hat{J}_{n(r)\mu}\left(m+\frac{n(r)}{\rho(\sigma)}\right) \quad (5.14c)$$

$$\hat{\bar{J}}_{n(r)\pm\rho(\sigma),\mu}\left(m\mp 1+\frac{n(r)\pm\rho(\sigma)}{\rho(\sigma)}\right) = \hat{\bar{J}}_{n(r)\mu}\left(m+\frac{n(r)}{\rho(\sigma)}\right). \quad (5.14d)$$

The periodicity of the twisted current modes in (5.14c), (5.14d) follows from the periodicity (5.3) of the twisted currents.

Then the orbifold OPEs give the *general twisted current algebra*

$$\begin{aligned}
&\left[\hat{J}_{n(r)\mu}\left(m+\frac{n(r)}{\rho(\sigma)}\right), \hat{J}_{n(s)\nu}\left(n+\frac{n(s)}{\rho(\sigma)}\right)\right] = \\
&\quad = i\mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(r)+n(s),\delta}(\sigma)\hat{J}_{n(r)+n(s),\delta}\left(m+n+\frac{n(r)+n(s)}{\rho(\sigma)}\right) + \\
&\quad + \left(m+\frac{n(r)}{\rho(\sigma)}\right)\delta_{m+n+\frac{n(r)+n(s)}{\rho(\sigma)},0}\mathcal{G}_{n(r)\mu;-n(r),\nu}(\sigma) \qquad (5.15a)
\end{aligned}$$

$$
\begin{aligned}
&\left[\hat{\bar{J}}_{n(r)\mu}\left(m+\frac{n(r)}{\rho(\sigma)}\right),\hat{\bar{J}}_{n(s)\nu}\left(n+\frac{n(s)}{\rho(\sigma)}\right)\right]= \\
&\qquad = i\mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(r)+n(s),\delta}(\sigma)\hat{\bar{J}}_{n(r)+n(s),\delta}\left(m+n+\frac{n(r)+n(s)}{\rho(\sigma)}\right)- \\
&\qquad\quad -\left(m+\frac{n(r)}{\rho(\sigma)}\right)\delta_{m+n+\frac{n(r)+n(s)}{\rho(\sigma)},0}\mathcal{G}_{n(r)\mu;-n(r),\nu}(\sigma) \qquad (5.15\text{b}) \\
&\left[\hat{J}_{n(r)\mu}\left(m+\frac{n(r)}{\rho(\sigma)}\right),\hat{\bar{J}}_{n(s)\nu}\left(n+\frac{n(s)}{\rho(\sigma)}\right)\right]=0 \\
&m,n\in\mathbb{Z}\,,\quad \sigma=0,\ldots,N_c-1 \qquad (5.15\text{c})
\end{aligned}
$$

in each sector σ of all WZW orbifolds $A_g(H)/H$. The general twisted current algebra (5.15) is a central result of this paper.

We have checked that the orbifold adjoint [32] of the twisted currents

$$
\hat{J}_{n(r)\mu}\left(m+\frac{n(r)}{\rho(\sigma)}\right)^{\dagger}=\sum_{\nu}\mathcal{R}_{n(r)\mu}{}^{-n(r),\nu}(\sigma)\hat{J}_{-n(r),\nu}\left(-m-\frac{n(r)}{\rho(\sigma)}\right) \qquad (5.16\text{a})
$$

$$
\hat{\bar{J}}_{n(r)\mu}\left(m+\frac{n(r)}{\rho(\sigma)}\right)^{\dagger}=\sum_{\nu}\mathcal{R}_{n(r)\mu}{}^{-n(r),\nu}(\sigma)\hat{\bar{J}}_{-n(r),\nu}\left(-m-\frac{n(r)}{\rho(\sigma)}\right) \qquad (5.16\text{b})
$$

defines a real form of the general twisted current algebra (5.15). Here, $\mathcal{R}$ is the orbifold conjugation matrix of ref. [32].

The integral affine subalgebra of the general twisted current algebra (5.15) is

$$
\begin{aligned}
[\hat{J}_{0\mu}(m),\hat{J}_{0\nu}(n)] &= i\mathcal{F}_{0\mu;0\nu}{}^{0\delta}(\sigma)\hat{J}_{0\delta}(m+n)+m\delta_{m+n,0}\mathcal{G}_{0\mu;0\nu}(\sigma) && (5.17\text{a})\\
[\hat{\bar{J}}_{0\mu}(m),\hat{\bar{J}}_{0\nu}(n)] &= i\mathcal{F}_{0\mu;0\nu}{}^{0\delta}(\sigma)\hat{\bar{J}}_{0\delta}(m+n)-m\delta_{m+n,0}\mathcal{G}_{0\mu;0\nu}(\sigma)\,, && (5.17\text{b})\\
[\hat{J}_{0\mu}(m),\hat{\bar{J}}_{0\nu}(0)] &= 0\,. && (5.17\text{c})
\end{aligned}
$$

We also note the Lie subalgebra

$$
\begin{aligned}
&[\hat{J}_{0\mu}(0),\hat{J}_{0\nu}(0)]=i\mathcal{F}_{0\mu;0\nu}{}^{0\delta}(\sigma)\hat{J}_{0\delta}(0)\,,\quad [\hat{\bar{J}}_{0\mu}(0),\hat{\bar{J}}_{0\nu}(0)]=i\mathcal{F}_{0\mu;0\nu}{}^{0\delta}(\sigma)\hat{\bar{J}}_{0\delta}(0)\\
&[\hat{J}_{0\mu}(0),\hat{\bar{J}}_{0\nu}(0)]=0 \qquad (5.18)
\end{aligned}
$$

generated by the zero modes of the integral affine subalgebra. It is expected (see sections 8 and 10) that these zero modes contribute to the *residual symmetry* in each sector σ of each WZW orbifold $A_g(H)/H$. Further discussion of the general twisted current algebra is found in the following subsection, and special cases are worked out in sections 7 and 10.

The twisted affine-Sugawara generators $L_\sigma(m)$, $\bar{L}_\sigma(m)$ satisfy the algebra of Vir$\oplus$Vir, and we also find the commutators

$$
[L_\sigma(m),\hat{J}_{n(r)\mu}(z,\sigma)] = z^m(z\partial+(m+1))\hat{J}_{n(r)\mu}(z,\sigma) \qquad (5.19\text{a})
$$

$$[\bar{L}_\sigma(m), \hat{\bar{J}}_{n(r)\mu}(\bar{z},\sigma)] = \bar{z}^m(\bar{z}\bar{\partial} + (m+1))\hat{\bar{J}}_{n(r)\mu}(\bar{z},\sigma) \tag{5.19b}$$

$$\left[L_\sigma(m), \hat{J}_{n(r)\mu}\left(n+\frac{n(r)}{\rho(\sigma)}\right)\right] = -\left(n+\frac{n(r)}{\rho(\sigma)}\right)\hat{J}_{n(r)\mu}\left(n+m+\frac{n(r)}{\rho(\sigma)}\right) \tag{5.19c}$$

$$\left[\bar{L}_\sigma(m), \hat{\bar{J}}_{n(r)\mu}\left(n+\frac{n(r)}{\rho(\sigma)}\right)\right] = \left(n+\frac{n(r)}{\rho(\sigma)}\right)\hat{\bar{J}}_{n(r)\mu}\left(n-m+\frac{n(r)}{\rho(\sigma)}\right) \tag{5.19d}$$

and the right mover modes commute with the left mover modes.

Finally, we also obtain the commutators with the twisted affine primary fields

$$\left[\hat{J}_{n(r)\mu}\left(m+\frac{n(r)}{\rho(\sigma)}\right), \hat{g}(\mathcal{T},\bar{z},z,\sigma)\right] = \hat{g}(\mathcal{T},\bar{z},z,\sigma)\mathcal{T}_{n(r)\mu}(T,\sigma)z^{m+\frac{n(r)}{\rho(\sigma)}} \tag{5.20a}$$

$$\left[\hat{\bar{J}}_{n(r)\mu}\left(m+\frac{n(r)}{\rho(\sigma)}\right), \hat{g}(\mathcal{T},\bar{z},z,\sigma)\right] = -\bar{z}^{-(m+\frac{n(r)}{\rho(\sigma)})}\mathcal{T}_{n(r)\mu}(T,\sigma)\hat{g}(\mathcal{T},\bar{z},z,\sigma) \tag{5.20b}$$

$$[L_\sigma(m), \hat{g}(\mathcal{T},\bar{z},z,\sigma)] = \hat{g}(\mathcal{T},\bar{z},z,\sigma)(\overleftarrow{\partial}z + (m+1)\mathcal{D}_{\hat{\mathfrak{g}}(\sigma)}(\mathcal{T}))z^m \tag{5.20c}$$

$$[\bar{L}_\sigma(m), \hat{g}(\mathcal{T},\bar{z},z,\sigma)] = \bar{z}^m(\bar{z}\bar{\partial} + (m+1)\mathcal{D}_{\hat{\mathfrak{g}}(\sigma)}(\mathcal{T}))\hat{g}(\mathcal{T},\bar{z},z,\sigma) \tag{5.20d}$$

and we have checked all Jacobi identities among the commutators (5.15), (5.19) and (5.20).

5.4 Identification of the general twisted current algebra

In this subsection, we discuss the form of the general twisted current algebra (5.15).

The general twisted left mover current algebra (5.15a) was called $\hat{\mathfrak{g}}(\sigma) = \hat{\mathfrak{g}}(H \subset \mathrm{Aut}(g); \sigma)$ in refs. [31] and [32]. The twisted right mover algebra in (5.15b) appears for the first time in this paper. The twisted right mover algebra has the same form as the twisted left mover algebra, but with the opposite sign of the central term. Both algebras are consistent with the Jacobi identity [32], since the $\mathcal{F}$ and $\mathcal{G}$ terms satisfy the identity separately. We will find it convenient to refer to the general twisted current algebra (5.15) as

$$\hat{\mathfrak{g}}(h_\sigma) \oplus \hat{\bar{\mathfrak{g}}}(h_\sigma), \qquad \hat{\mathfrak{g}}(h_\sigma) \equiv \hat{\mathfrak{g}}(H \subset \mathrm{Aut}(g); h_\sigma) \equiv \hat{\mathfrak{g}}(H \subset \mathrm{Aut}(g); \sigma) \tag{5.21}$$

and more generally to follow the convention $\sigma \to h_\sigma$ in the discussion below.

In fact the twisted right mover algebra of sector σ is not a new algebra. To understand this, consider first the mode-number reversed form of this algebra

$$\left[\hat{\bar{J}}^{\mathrm{R}}_{n(r)\mu}\left(m+\frac{n(r)}{\rho(\sigma)}\right), \hat{\bar{J}}^{\mathrm{R}}_{n(s)\nu}\left(n+\frac{n(s)}{\rho(\sigma)}\right)\right] = i\mathcal{F}_{-n(r),\mu;-n(s),\nu}{}^{-n(r)-n(s),\delta}(\sigma) \times$$
$$\times \hat{\bar{J}}^{\mathrm{R}}_{n(r)+n(s),\delta}\left(m+n+\frac{n(r)+n(s)}{\rho(\sigma)}\right) +$$

$$
\begin{aligned}
&+\left(m+\frac{n(r)}{\rho(\sigma)}\right)\delta_{m+n+\frac{n(r)+n(s)}{\rho(\sigma)},0}\times \\
&\times\mathcal{G}_{-n(r),\mu;n(r),\nu}(\sigma) \qquad (5.22a)
\end{aligned}
$$

$$
\hat{\bar{J}}^{\mathrm{R}}_{n(r)\mu}\left(m+\frac{n(r)}{\rho(\sigma)}\right)\equiv\hat{\bar{J}}_{-n(r),\mu}\left(-m-\frac{n(r)}{\rho(\sigma)}\right) \qquad (5.22b)
$$

which exhibits the correct sign of the central term.

We claim that the algebra (5.22) is isomorphic to the twisted left mover current algebra of the sector corresponding to $h_\sigma^{-1}\in H$

$$
\hat{\bar{\mathfrak{g}}}(h_\sigma)\simeq\hat{\mathfrak{g}}(h_\sigma^{-1})\,. \qquad (5.23)
$$

To see this, one needs to compare the corresponding forms of the H-eigenvalue problem

$$
\begin{aligned}
w(h_\sigma^{-1}) &= w^\dagger(h_\sigma),\quad \rho(h_\sigma^{-1})=\rho(h_\sigma) \qquad (5.24a)\\
w(h_\sigma)U^\dagger(h_\sigma) &= U^\dagger(h_\sigma)E_{n(r)}(h_\sigma)\\
w(h_\sigma^{-1})U^\dagger(h_\sigma) &= w^\dagger(h_\sigma)U^\dagger(h_\sigma)=U^\dagger(h_\sigma)E_{n(r)}(h_\sigma)^*\,, \qquad (5.24b)
\end{aligned}
$$

where we used (3.13) to obtain (5.24b). This allows us to choose

$$
U^\dagger(h_\sigma^{-1})_a{}^{n(r)\mu}=U^\dagger(h_\sigma)_a{}^{-n(r),\mu} \qquad (5.25)
$$

and then the duality transformations for $\mathcal{G}(\sigma)$ and $\mathcal{F}(\sigma)$ in (4.7) imply that

$$
\begin{aligned}
\mathcal{G}_{n(r)\mu;n(s)\nu}(h_\sigma^{-1}) &= \mathcal{G}_{-n(r),\mu;-n(s),\nu}(h_\sigma),\ \mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(t)\delta}(h_\sigma^{-1})\\
&= \mathcal{F}_{-n(r),\mu;-n(s),\nu}{}^{-n(t),\delta}(h_\sigma) \qquad (5.26a)\\
\left\{\hat{\bar{J}}^{\mathrm{R}}_{n(r)\mu}\left(m+\frac{n(r)}{\rho(\sigma)}\right),\hat{\bar{J}}^{\mathrm{R}}_{n(s)\nu}\left(m+\frac{n(r)}{\rho(\sigma)}\right)\right\} &= i\mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(r)+n(s),\delta}(h_\sigma^{-1})\hat{\bar{J}}^{\mathrm{R}}_{n(r)+n(s),\delta}\times\\
&\quad\times\left(m+n+\frac{n(r)+n(s)}{\rho(\sigma)}\right)+\\
&\quad+\left(m+\frac{n(r)}{\rho(\sigma)}\right)\delta_{m+n+\frac{n(r)+n(s)}{\rho(\sigma)},0}\times\\
&\quad\times\mathcal{G}_{n(r)\mu;-n(r),\nu}(h_\sigma^{-1})
\end{aligned}
\qquad (5.26b)
$$

which establishes the claim. This argument identifies the general twisted current algebra (5.15) as

$$
\mathrm{affine}(\mathrm{g}\oplus\mathrm{g})\underset{\sigma}{\longrightarrow}\hat{\mathfrak{g}}(\mathrm{h}_\sigma)\oplus\hat{\bar{\mathfrak{g}}}(\mathrm{h}_\sigma)\simeq\hat{\mathfrak{g}}(\mathrm{h}_\sigma)\oplus\hat{\mathfrak{g}}(\mathrm{h}_\sigma^{-1}) \qquad (5.27)
$$

in sector σ of $A_g(H)/H$.

Although the form (5.15) of the general twisted current algebra is new, it is important to note that this form is in fact consistent with what is known about partition functions and characters of general orbifolds [25]. The point is that when one sees

$$Z = \sum_{\sigma,i} |\chi_i(q,\sigma)|^2 \tag{5.28}$$

in the literature, it is supposed to be read as

$$Z = \sum_{\sigma,i} \chi_i(q,\sigma)\chi_{i^\vee}(\bar{q},\sigma)\,, \tag{5.29}$$

where $i^\vee$ denotes the charge conjugate field of i. Moreover, it is known that when i is a representation in the sector twisted by h_σ then $i^\vee$ is a representation in the sector twisted by h_σ^{-1} (so that the two sectors can combine to the identity). Thus, in our notation, what is required by (5.29)

$$Z = \sum_{\sigma,i} \chi_i(q,h_\sigma)\chi_{i^\vee}(\bar{q},h_\sigma^{-1}) \tag{5.30}$$

is in precise agreement with the identification $\hat{\mathfrak{g}}(h_\sigma) \oplus \hat{\mathfrak{g}}(h_\sigma^{-1})$ of the general twisted current algebra (5.15).

We also point out that the general twisted current algebra (5.15) guarantees the consistency of the equations for general twisted boundary states

$$\left(\hat{J}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)}) + \hat{\bar{J}}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)})\right)|B\rangle_\sigma = 0, \qquad \sigma = 0,\ldots,N_c-1 \tag{5.31}$$

because the sum of the twisted left and right mover currents has no central term.

5.5 The rectification problem

The identification (5.27) of the general twisted current algebra leaves open the question of when the twisted right and left mover current algebras are copies

$$\hat{\mathfrak{g}}(h_\sigma^{-1}) \simeq \hat{\mathfrak{g}}(h_\sigma) \tag{5.32}$$

which we call *the rectification problem.*

In a simpler example, we find for all constants $\theta(\sigma)$ that the further redefinition

$$\hat{\bar{J}}^\sharp_{n(r)\mu}\left(m+\frac{n(r)}{\rho(\sigma)}\right) \equiv \theta_{n(r)\mu}(\sigma)\hat{\bar{J}}_{-n(r),\mu}\left(-m-\frac{n(r)}{\rho(\sigma)}\right) \tag{5.33a}$$

$$\left[\bar{L}_\sigma(m), \hat{\bar{J}}^\sharp_{n(r)\mu}\left(n+\frac{n(r)}{\rho(\sigma)}\right)\right] = -\left(n+\frac{n(r)}{\rho(\sigma)}\right)\hat{\bar{J}}^\sharp_{n(r)\mu}\left(n+m+\frac{n(r)}{\rho(\sigma)}\right) \tag{5.33b}$$

$$\hat{\bar{J}}_{n(r)\mu}(\bar{z},\sigma) = \theta^{-1}_{-n(r),\mu}(\sigma)\sum_{m\in\mathbb{Z}}\hat{\bar{J}}^\sharp_{-n(r),\mu}\left(m-\frac{n(r)}{\rho(\sigma)}\right)\bar{z}^{-(m-\frac{n(r)}{\rho(\sigma)})-1} \tag{5.33c}$$

rectifies the right mover commutator (5.19d) into a copy (5.33b) of the left mover commutator (5.19c). In the language of refs. [35, 36], the commutators (5.19c) and (5.33b) identify the currents $\hat{J}$ and $\hat{\bar{J}}^{\sharp}$

$$\hat{J}_{n(r)\mu}(z,\sigma) = \sum_{m\in\mathbb{Z}} \hat{J}_{n(r)\mu}\left(m+\frac{n(r)}{\rho(\sigma)}\right) z^{-\left(m+\frac{n(r)}{\rho(\sigma)}\right)-1} \tag{5.34a}$$

$$\hat{\bar{J}}^{\sharp}_{n(r)\mu}(\bar{z},\sigma) \equiv \sum_{m\in\mathbb{Z}} \hat{\bar{J}}^{\sharp}_{n(r)\mu}\left(m+\frac{n(r)}{\rho(\sigma)}\right) \bar{z}^{-\left(m+\frac{n(r)}{\rho(\sigma)}\right)-1} = \theta_{n(r)\mu}(\sigma)\hat{\bar{J}}_{-n(r),\mu}(\bar{z},\sigma) \tag{5.34b}$$

$$\hat{J}_{n(r)\mu}(ze^{2\pi i}) = e^{-2\pi i\frac{n(r)}{\rho(\sigma)}}\hat{J}_{n(r)\mu}(z)\,, \qquad \hat{\bar{J}}^{\sharp}_{n(r)\mu}(\bar{z}e^{-2\pi i}) = e^{2\pi i\frac{n(r)}{\rho(\sigma)}}\hat{\bar{J}}^{\sharp}_{n(r)\mu}(\bar{z}) \tag{5.34c}$$

as twisted (1,0) and twisted (0,1) operators respectively under the left and right mover twisted affine-Sugawara constructions ($\hat{\bar{T}}\hat{\bar{J}}^{\sharp}$ is a barred copy of (5.11a)).

For the current algebra itself, the rectification problem may be trivially solved by $\theta(0) = 1$ in the untwisted sector $\sigma = 0$ of any orbifold

$$U(0) = 1\,,\quad \mathcal{G}(0) = G\,,\quad \mathcal{F}(0) = f\,,\quad J(m) \equiv \hat{J}(m)\,,\quad \bar{J}^{\sharp}(m) \equiv \hat{\bar{J}}(-m) \tag{5.35a}$$

$$[J_a(m), J_b(n)] = if_{ab}{}^{c}J_c(m+n) + mG_{ab}\delta_{m+n,0} \tag{5.35b}$$

$$[\bar{J}^{\sharp}_a(m), \bar{J}^{\sharp}_b(n)] = if_{ab}{}^{c}\bar{J}^{\sharp}_c(m+n) + mG_{ab}\delta_{m+n,0}\,, \qquad [J_a(m), \bar{J}^{\sharp}_b(n)] = 0 \tag{5.35c}$$

because $h_0 = h_0^{-1} = 1$.

So far as the general twisted current algebra is concerned, the situation does not seem so simple: We shall find for the permutation orbifolds (see subsections 7.3, 7.4 and 7.5) and the inner-automorphic orbifolds (see subsection 10.1) that it is possible for all σ to choose the constants $\theta(\sigma)$ to rectify the right mover algebra $\hat{\mathfrak{g}}(h_\sigma^{-1})$ into a copy $\hat{\mathfrak{g}}(h_\sigma)$ of the left mover algebra. However, we do not have a proof[2] that such rectification can be extended across all twisted right mover algebras $\hat{\mathfrak{g}}(h_\sigma^{-1})$.

5.6 Orbifold correlators and $L_\sigma(0)$, $\bar{L}_\sigma(0)$ ward identities

For all WZW orbifolds $A_g(H)/H$, the correlators of the twisted primary fields are

$$\begin{aligned}\hat{A}(\sigma) \equiv \hat{A}_\sigma(\mathcal{T},\bar{z},z) &\equiv {}_\sigma\langle 0|\hat{g}(\mathcal{T}^{(1)},\bar{z}_1,z_1,\sigma)\cdots\hat{g}(\mathcal{T}^{(N)},\bar{z}_N,z_N,\sigma)|0\rangle_\sigma \\ &= \langle\hat{g}(\mathcal{T}^{(1)},\bar{z}_1,z_1,\sigma)\cdots\hat{g}(\mathcal{T}^{(N)},\bar{z}_N,z_N,\sigma)\rangle_\sigma\end{aligned} \tag{5.36}$$

and in what follows, we use the notation[3]

$$T^{(\mu)}, \bar{z}_\mu, z_\mu, \qquad \mu = 1\ldots N \quad : \quad [\mathcal{T}^{(\mu)}, \mathcal{T}^{(\nu)}] = 0 \quad \text{when } \mu \neq \nu \tag{5.37}$$

[2]One may show however that a sufficient condition for the rectification of the twisted right mover current algebra is that we can find a g automorphism $A(\sigma)$ which relates the action $w(h_\sigma^{-1}) = w^\dagger(h_\sigma)$ to the action $w(h_\sigma)$.

[3]The distinction between these Greek letters and those of the degeneracy indices will be clear in context.

for the coordinates and twisted representation matrices. In (5.36), the ground state $|0\rangle_\sigma$ (or twist field state[4]) of sector σ is primary under Vir$\oplus$Vir

$$L_\sigma(m \geq 0)|0\rangle_\sigma = \delta_{m,0}\hat{\Delta}_0(\sigma)|0\rangle_\sigma\,, \quad \bar{L}_\sigma(m \geq 0)|0\rangle_\sigma = \delta_{m,0}\hat{\bar{\Delta}}_0(\sigma)|0\rangle_\sigma \quad (5.38\text{a})$$
$$_\sigma\langle 0|L_\sigma(m \leq 0) = {}_\sigma\langle 0|\hat{\Delta}_0(\sigma)\delta_{m,0}\,, \quad _\sigma\langle 0|\bar{L}_\sigma(m \leq 0) = {}_\sigma\langle 0|\hat{\bar{\Delta}}_0(\sigma)\delta_{m,0}\,. \quad (5.38\text{b})$$

Using (5.20c), (5.20d), this gives the $L_\sigma(0)$, $\bar{L}_\sigma(0)$ Ward identities

$$\hat{A}(\sigma)\sum_{\mu=1}^{N}\left(\overleftarrow{\partial}_\mu z_\mu + \mathcal{D}_{\hat{\mathfrak{g}}(\sigma)}(\mathcal{T}^{(\mu)})\right) = 0, \qquad \sum_{\mu=1}^{N}\left(\bar{z}_\mu\bar{\partial}_\mu + \mathcal{D}_{\hat{\mathfrak{g}}(\sigma)}(\mathcal{T}^{(\mu)})\right)\hat{A}(\sigma) = 0\,, \quad (5.39)$$

where $\mathcal{D}_{\hat{\mathfrak{g}}(\sigma)}(\mathcal{T}^{(\mu)})$ is the twisted conformal weight matrix (4.26) of twisted representation $\mathcal{T}^{(\mu)}$. Other Ward identities exist which are associated to the Lie subalgebra (5.18) and other subalgebras of the general twisted current algebra, and we discuss these on a case by case basis below. We also note that many orbifolds have other Ward identities associated to extended Virasoro algebras (see ref. [33]) but we will not study these here.

5.7 The classical theory of WZW orbifolds

We finally consider the corresponding classical theory of all WZW orbifolds, where we find that locality dictates the monodromies (5.5) of the twisted right and left mover currents.

In the classical WZW model on semisimple g, we know that the matrix currents have the local form

$$J(T,z) = J_a(z)G^{ab}T_b \propto g^{-1}(T,\bar{z},z)\bar{\partial}g(T,\bar{z},z) \quad (5.40\text{a})$$
$$\bar{J}(T,\bar{z}) = \bar{J}_a(z)G^{ab}T_b \propto g(T,\bar{z},z)\partial g^{-1}(T,\bar{z},z) \quad (5.40\text{b})$$
$$g^\dagger(T,\bar{z},z)g(T,\bar{z},z) = 1\,, \qquad \bar{\partial}J(T,z) = \partial\bar{J}(T,\bar{z}) = 0\,, \quad (5.40\text{c})$$

where $g(T,\bar{z},z)$ is the *group element* in matrix irrep T. We note in particular that the linkage relation (3.5) guarantees that an H-transformation of a group element is still a group element

$$g(T,\bar{z},z) = e^{i\beta^a(\bar{z},z)T_a} \quad (5.41\text{a})$$
$$g(T,\bar{z},z)' = W(h_\sigma;T)g(T,\bar{z},z)W^\dagger(h_\sigma;T) = e^{i\beta^a(\bar{z},z)W(h_\sigma;T)T_aW^\dagger(h_\sigma;T)} = e^{i\beta^a(\bar{z},z)'T_a} \quad (5.41\text{b})$$
$$\beta^a(\bar{z},z)' \equiv \beta^b(\bar{z},z)w^\dagger(h_\sigma)_b{}^a\,, \quad (5.41\text{c})$$

where the linkage relation in the form (3.5a) was used to obtain the final form.

[4]As seen in (5.38), twist field states $|0\rangle_\sigma = \tau_\sigma(0)|0\rangle$ are not in general $Sl(2)\oplus Sl(2)$ invariant so that translation invariance is lost [33] in the twisted sectors.

The group elements $g(T,\bar z,z)$ are the classical limit [36] of the affine primary fields, so, following (4.2), we define the *eigengroup elements* in terms of the group elements as

$$\mathcal{G}(\mathcal{T},\bar z,z,\sigma)\equiv U(T,\sigma)g(T,\bar z,z)U^\dagger(T,\sigma)=e^{i\mathfrak{b}^{n(r)\mu}(\bar z,z,\sigma)\mathcal{T}_{n(r)\mu}(T,\sigma)} \tag{5.42a}$$

$$\mathcal{G}^\dagger(\mathcal{T},\bar z,z,\sigma)\mathcal{G}(\mathcal{T},\bar z,z,\sigma)=\mathbb{1},\qquad \mathfrak{b}^{n(r)\mu}(\bar z,z,\sigma)=\chi(\sigma)^{-1}_{n(r)\mu}\,\beta^a(\bar z,z)\,U^\dagger(\sigma)_a{}^{n(r)\mu} \tag{5.42b}$$

$$\begin{aligned}&\mathcal{G}(\mathcal{T},\bar z,z,\sigma)'=E(T,\sigma)\mathcal{G}(\mathcal{T},\bar z,z,\sigma)E(T,\sigma)^*\,,\\ &\mathfrak{b}^{n(r)\mu}(\bar z,z,\sigma)'=\mathfrak{b}^{n(r)\mu}(\bar z,z,\sigma)E_{n(r)}(\sigma)^*\,,\end{aligned} \tag{5.42c}$$

where $\{\mathfrak{b}^{n(r)\mu}\}$ are the tangent space eigencoordinates. With the selection rules (4.20) of the twisted representation matrices $\mathcal{T}$, the equivalence of the automorphic responses in (5.42c) is easily checked to all orders in $\mathfrak{b}$.

Then by local isomorphisms we obtain the classical twisted matrix currents and the classical *group orbifold elements* $\hat g(\mathcal{T},\bar z,z,\sigma)$

$$\mathcal{J}_{n(r)\mu}(z,\sigma)\underset{\sigma}{\longrightarrow}\hat J_{n(r)\mu}(z,\sigma),\qquad \bar{\mathcal{J}}_{n(r)\mu}(\bar z,\sigma)\underset{\sigma}{\longrightarrow}\hat{\bar J}_{n(r)\mu}(\bar z,\sigma) \tag{5.43a}$$

$$\mathcal{G}(\mathcal{T},\bar z,z,\sigma)\underset{\sigma}{\longrightarrow}\hat g(\mathcal{T},\bar z,z,\sigma),\qquad \mathcal{G}^{-1}(\mathcal{T},\bar z,z,\sigma)\underset{\sigma}{\longrightarrow}\hat g^{-1}(\mathcal{T},\bar z,z,\sigma) \tag{5.43b}$$

$$\hat J(\mathcal{T},z,\sigma)=\hat J_{n(r)\mu}(z,\sigma)\mathcal{G}^{n(r)\mu;-n(r),\nu}(\sigma)\mathcal{T}_{-n(r),\nu}(T,\sigma)\propto\hat g^{-1}(\mathcal{T},\bar z,z,\sigma)\bar\partial\hat g(\mathcal{T},\bar z,z,\sigma) \tag{5.43c}$$

$$\hat{\bar J}(\mathcal{T},\bar z,\sigma)=\hat{\bar J}_{n(r)\mu}(\bar z,\sigma)\mathcal{G}^{n(r)\mu;-n(r),\nu}(\sigma)\mathcal{T}_{-n(r),\nu}(T,\sigma)\propto\hat g(\mathcal{T},\bar z,z,\sigma)\partial\hat g^{-1}(\mathcal{T},\bar z,z,\sigma) \tag{5.43d}$$

$$\hat g^\dagger(\mathcal{T},\bar z,z,\sigma)\hat g(\mathcal{T},\bar z,z,\sigma)=\mathbb{1},\qquad \bar\partial\hat J(\mathcal{T},z,\sigma)=\partial\hat{\bar J}(\mathcal{T},\bar z,\sigma)=0\,. \tag{5.43e}$$

The group orbifold elements are the classical limit of the twisted affine primary fields. Because the classical group elements have trivial monodromy, the local isomorphisms also give the *classical monodromies* of the group orbifold elements

$$\hat g(\mathcal{T},\bar z e^{-2\pi i},ze^{2\pi i},\sigma)=E(T,\sigma)\hat g(\mathcal{T},\bar z,z,\sigma)E(T,\sigma)^* \tag{5.44a}$$

$$\hat g(\mathcal{T},\bar z e^{-2\pi i},ze^{2\pi i},\sigma)_{N(r)\mu}{}^{N(s)\nu}=e^{-2\pi i\frac{N(r)-N(s)}{R(\sigma)}}\hat g(\mathcal{T},\bar z,z,\sigma)_{N(r)\mu}{}^{N(s)\nu} \tag{5.44b}$$

as the old automorphic responses (5.42c) of the eigengroup elements. The monodromy of $\hat g^{-1}$ is the same as $\hat g$.

Using local isomorphisms also for the tangent space eigencoordinates, the group orbifold elements can be considered as the exponentiation of the Lie algebra (4.12) of the twisted representation matrices

$$\mathfrak{b}^{n(r)\mu}\underset{\sigma}{\longrightarrow}\hat\beta^{n(r)\mu},\qquad \hat g(\mathcal{T},\bar z,z,\sigma)=e^{i\hat\beta^{n(r)\mu}(\bar z,z,\sigma)\mathcal{T}_{n(r)\mu}(T,\sigma)} \tag{5.45a}$$

$$\hat\beta^{n(r)\mu}(\bar z e^{-2\pi i},ze^{2\pi i},\sigma)=\hat\beta^{n(r)\mu}(\bar z,z,\sigma)e^{2\pi i\frac{n(r)}{\rho(\sigma)}}\,, \tag{5.45b}$$

where the quantities $\{\hat{\beta}^{n(r)\mu}\}$ are called the twisted tangent space coordinates. The selection rules (4.20) for $\mathcal{T}$ are again necessary to check the consistency of the monodromies (5.45b) and (5.44b). We also note that the multiplication law

$$\hat{g}_3(\mathcal{T},\bar{z},z,\sigma) \equiv \hat{g}_1(\mathcal{T},\bar{z},z,\sigma)\hat{g}_2(\mathcal{T},\bar{z},z,\sigma) \tag{5.46a}$$

$$\hat{\beta}_3(\bar{z},z,\sigma) = \hat{\beta}_1(\bar{z},z,\sigma)+\hat{\beta}_2(\bar{z},z,\sigma)+\mathcal{O}(\hat{\beta}^2) \tag{5.46b}$$

$$\hat{g}_i(\mathcal{T},\bar{z}e^{-2\pi i},ze^{2\pi i},\sigma) = E(T,\sigma)\hat{g}_i(\mathcal{T},\bar{z},z,\sigma)E(T,\sigma)^*, \qquad i=1,2,3 \tag{5.46c}$$

preserves the monodromies of the group orbifold elements.

We turn now to the monodromies of the twisted currents. These are easily obtained as above by local isomorphisms, but we wish to study these monodromies as they are dictated by the *local* relations (5.43c), (5.43d) in the classical theory. Using these relations and the monodromies (5.44) of the group orbifold elements, we find that the monodromies of the twisted matrix currents are the same

$$\hat{J}(\mathcal{T},ze^{2\pi i},\sigma)_{N(r)\mu}{}^{N(s)\nu} = e^{-2\pi i\frac{N(r)-N(s)}{R(\sigma)}}\hat{J}(\mathcal{T},z,\sigma)_{N(r)\mu}{}^{N(s)\nu} \tag{5.47a}$$

$$\hat{\bar{J}}(\mathcal{T},\bar{z}e^{-2\pi i},\sigma)_{N(r)\mu}{}^{N(s)\nu} = e^{-2\pi i\frac{N(r)-N(s)}{R(\sigma)}}\hat{\bar{J}}(\mathcal{T},z,\sigma)_{N(r)\mu}{}^{N(s)\nu} \tag{5.47b}$$

when the same path is followed for z and $\bar{z}=z^*$. Moreover, the selection rule (4.20b) for the twisted representation matrices

$$e^{-2\pi i\frac{n(r)}{\rho(\sigma)}}\mathcal{T}_{-n(r),\mu}(T,\sigma)_{N(s)\nu}{}^{N(t)\delta} = \mathcal{T}_{-n(r),\mu}(T,\sigma)_{N(s)\nu}{}^{N(t)\delta}e^{-2\pi i\frac{N(s)-N(t)}{R(\sigma)}} \tag{5.48}$$

shows that the monodromies (5.47) are equivalent to the monodromies

$$\hat{J}_{n(r)\mu}(ze^{2\pi i},\sigma) = e^{-2\pi i\frac{n(r)}{\rho(\sigma)}}\hat{J}_{n(r)\mu}(z,\sigma), \qquad \hat{\bar{J}}_{n(r)\mu}(\bar{z}e^{-2\pi i},\sigma) = e^{-2\pi i\frac{n(r)}{\rho(\sigma)}}\hat{\bar{J}}_{n(r)\mu}(\bar{z},\sigma) \tag{5.49}$$

of the twisted currents $\hat{J}_{n(r)\mu}$, $\hat{\bar{J}}_{n(r)\mu}$. As promised in Subsec. 5.1: The classical monodromies (5.49), which followed from the local classical relations (5.43c) and (5.43d), are in agreement with the quantum monodromies (5.5) of the twisted currents.

We can also obtain the corresponding set of WZW orbifold actions as follows. The general WZW action [37, 38] for semisimple g on the cylinder (t,ξ) with $0\le\xi\le 2\pi$ is

$$\begin{aligned} S_{WZW}[M,g] = &-\frac{1}{8\pi}\int d^2\xi\ \mathrm{Tr}\Big(M(k,T)\ g^{-1}(T)\partial_+g(T)\ g^{-1}(T)\partial_-g(T)\Big) - \\ &-\frac{1}{12\pi}\int_\Gamma \mathrm{Tr}\Big(M(k,T)\ (g^{-1}(T)dg(T))^3\ \Big), \quad d^2\xi\equiv dt\,d\xi \end{aligned} \tag{5.50a}$$

$$(g^{-1}dg)^3 = dt\,d\xi\,d\rho\,\epsilon^{ABC}(g^{-1}\partial_Ag)(g^{-1}\partial_Bg)(g^{-1}\partial_Cg), \quad \{A,B,C\}=\{t,\xi,\rho\} \tag{5.50b}$$

$$g(T)\equiv g(T,\xi,t), \qquad \partial_\pm\equiv\partial_t\pm\partial_\xi\,, \tag{5.50c}$$

where $\epsilon^{t\xi\rho} = 1$, $M(k,T)$ is the data matrix in (2.1) and $g(T)$ in the solid cylinder Γ is also a function of ρ. This form of the action provides a convenient packaging for the expected sum over the simple components of g,

$$\beta^a \;\to\; \beta^{aI}\,, \qquad g(T)_\alpha{}^\beta \;\to\; g(T)_{\alpha I}{}^{\beta J} = g_I(T^I)_{\alpha(I)}{}^{\beta(I)}\delta_I^J, \qquad g_I(T^I) = e^{i\sum_a \beta^{aI}T_a^I} \tag{5.51a}$$

$$S_{\rm WZW}[M,g] = -\frac{1}{8\pi}\sum_I \frac{k_I}{y_I(T^I)}\int d^2\xi\,{\rm Tr}\Big(g_I^{-1}(T^I)\partial_+ g_I(T^I)\; g_I^{-1}(T^I)\partial_- g_I(T^I)\Big) + \cdots \tag{5.51b}$$

where we have used (2.6), (2.7) and (5.41a). The automorphic responses of the group elements on the cylinder are given by (5.41) with $(\bar z, z) \to (\xi, t)$, so we find that the H-invariance of the general WZW action

$$S_{\rm WZW}[M,g'] = S_{\rm WZW}[M,g], \qquad g(T,\xi,t)' = W(h_\sigma;T)g(T,\xi,t)W^\dagger(h_\sigma;T) \tag{5.52}$$

is guaranteed by the H-invariance (3.10) of the data matrix.

Eigengroup elements on the cylinder are also defined as in (5.42) with $(\bar z, z) \to (\xi, t)$. Then local isomorphisms give the *general WZW orbifold action*

$$S_{\rm WZW}[\mathcal{M},\mathcal{G}] \equiv S_{\rm WZW}[M,g(\mathcal{G})] \underset{\sigma}{\longrightarrow} \hat S_\sigma[\mathcal{M},\hat g] \tag{5.53a}$$

$$\begin{aligned}\hat S_\sigma[\mathcal{M},\hat g] = &-\frac{1}{8\pi}\int d^2\xi\;\widehat{\rm Tr}\Big(\,\mathcal{M}(\mathcal{T},\sigma)\;\hat g^{-1}(\mathcal{T},\sigma)\partial_+\hat g(\mathcal{T},\sigma)\;\hat g^{-1}(\mathcal{T},\sigma)\partial_-\hat g(\mathcal{T},\sigma)\,\Big) - \\ &-\frac{1}{12\pi}\int_\Gamma \widehat{\rm Tr}\Big(\,\mathcal{M}(\mathcal{T},\sigma)\;(\,\hat g^{-1}(\mathcal{T},\sigma)d\hat g(\mathcal{T},\sigma)\,)^3\,\Big)\,, \\ &\sigma = 0,\ldots,N_c-1\end{aligned} \tag{5.53b}$$

$$(\hat g^{-1}d\hat g)^3 = dt\,d\xi\,d\rho\;\epsilon^{ABC}(\hat g^{-1}\partial_A\hat g)(\hat g^{-1}\partial_B\hat g)(\hat g^{-1}\partial_C\hat g), \quad \{A,B,C\} = \{t,\xi,\rho\} \tag{5.53c}$$

$$\hat g^\dagger(\mathcal{T},\xi,t,\sigma)\hat g(\mathcal{T},\xi,t,\sigma) = 1 \tag{5.53d}$$

$$\hat g(\mathcal{T},\xi+2\pi,t,\sigma)_{N(r)\mu}{}^{N(s)\nu} = e^{-2\pi i\frac{N(r)-N(s)}{R(\sigma)}}\hat g(\mathcal{T},\xi,t,\sigma)_{N(r)\mu}{}^{N(s)\nu} \tag{5.53e}$$

$$\hat S_\sigma[\mathcal{M}(\mathcal{T},\sigma),\hat g(\mathcal{T},\xi+2\pi,t,\sigma)] = \hat S_\sigma[\mathcal{M}(\mathcal{T},\sigma),\hat g(\mathcal{T},\xi,t,\sigma)] \tag{5.53f}$$

for sector σ of each WZW orbifold $A_g(H)/H$. Here, $\hat g(\mathcal{T},\sigma) \equiv \hat g(\mathcal{T},\xi,t,\sigma)$ is the group orbifold element on the cylinder[5], the orbifold trace $\widehat{\rm Tr}$ is defined in (4.15) and the quantity $\mathcal{M}(\mathcal{T},\sigma)$ is the twisted data matrix in (4.16). The monodromy of $\hat g^{-1}$ is the same as $\hat g$ in (5.53e). The trivial monodromy (5.53f) of the WZW orbifold action follows from the monodromy (5.53e) of the group orbifold elements and the selection rule (4.18) of the twisted data matrix.

[5]The automorphic response of the group element g and hence the ξ-monodromy of $\hat g$ is the same when the group orbifold elements are extended to the solid cylinder Γ.

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Because the twisted data matrix is invertible and commutes with $\hat{g}(\mathcal{T},\sigma)$ (see eq. (5.2)), the equations of motion of this action

$$\begin{aligned}\hat{J}(\mathcal{T},\xi,t,\sigma) &\equiv -\frac{i}{2}\hat{g}^{-1}(\mathcal{T},\xi,t,\sigma)\partial_+\hat{g}(\mathcal{T},\xi,t,\sigma)\\ &= \hat{J}_{n(r)\mu}(\xi,t,\sigma)\mathcal{G}^{n(r)\mu;-n(r),\nu}(\sigma)\mathcal{T}_{-n(r),\nu}(T,\sigma) \qquad (5.54a)\\ \hat{\bar{J}}(\mathcal{T},\xi,t,\sigma) &\equiv -\frac{i}{2}\hat{g}(\mathcal{T},\xi,t,\sigma)\partial_-\hat{g}^{-1}(\mathcal{T},\xi,t,\sigma)\\ &= \hat{\bar{J}}_{n(r)\mu}(\xi,t,\sigma)\mathcal{G}^{n(r)\mu;-n(r),\nu}(\sigma)\mathcal{T}_{-n(r),\nu}(T,\sigma) \qquad (5.54b)\\ \partial_-\hat{J}(\mathcal{T},\xi,t,\sigma) &= \partial_+\hat{\bar{J}}(\mathcal{T},\xi,t,\sigma) = 0 \qquad (5.54c)\end{aligned}$$

are equivalent to those found on the sphere in (5.40c). Moreover, following the development on the sphere, one finds the additional monodromies on the cylinder

$$\begin{aligned}\hat{g}(\mathcal{T},\xi,t,\sigma) &= e^{i\hat{\beta}^{n(r)\mu}(\xi,t,\sigma)\mathcal{T}_{n(r)\mu}(T,\sigma)},\\ \hat{\beta}^{n(r)\mu}(\xi+2\pi,t,\sigma) &= \hat{\beta}^{n(r)\mu}(\xi,t,\sigma)e^{2\pi i\frac{n(r)}{\rho(\sigma)}} \qquad (5.55a)\\ \hat{J}_{n(r)\mu}(\xi+2\pi,t,\sigma) &= e^{-2\pi i\frac{n(r)}{\rho(\sigma)}}\hat{J}_{n(r)\mu}(\xi,t,\sigma),\\ \hat{\bar{J}}_{n(r)\mu}(\xi+2\pi,t,\sigma) &= e^{-2\pi i\frac{n(r)}{\rho(\sigma)}}\hat{\bar{J}}_{n(r)\mu}(\xi,t,\sigma) \qquad (5.55b)\end{aligned}$$

which are also equivalent to those found on the sphere.

Further remarks on the classical theory of WZW orbifolds are found in subsections 7.5 and 10.5

6. Mode normal ordering

6.1 Exact operator products

In this subsection, we introduce a mode normal ordering [32] to discuss exact operator products of the twisted currents with themselves and with the twisted affine primary fields. Moreover, we will follow the usual [36] right-left mover factorization

$$\hat{g}(\mathcal{T},\bar{z},z,\sigma) = \hat{g}_-(\mathcal{T},\bar{z},\sigma)\hat{g}_+(\mathcal{T},z,\sigma), \qquad \hat{A}(\sigma) = \hat{A}_-(\sigma)\hat{A}_+(\sigma) \qquad (6.1a)$$

$$\left[\hat{\bar{J}}_{n(r)\mu}\left(m+\frac{n(r)}{\rho(\sigma)}\right), \hat{g}_+(\mathcal{T},z,\sigma)\right] = 0,$$

$$\left[\hat{J}_{n(r)\mu}\left(m+\frac{n(r)}{\rho(\sigma)}\right), \hat{g}_-(\mathcal{T},\bar{z},\sigma)\right] = 0 \qquad (6.1b)$$

$$\partial\hat{g}_+(\mathcal{T},z,\sigma) = 2\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)\mu;-n(r),\nu}(\sigma) : \hat{J}_{n(r)\mu}(z,\sigma)\hat{g}_+(\mathcal{T},z,\sigma)\mathcal{T}_{-n(r),\nu} : \qquad (6.1c)$$

$$\bar{\partial}\hat{g}_-(\mathcal{T},\bar{z},\sigma) = -2\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)\mu;-n(r),\nu}(\sigma) : \hat{\bar{J}}_{n(r)\mu}(\bar{z},\sigma)\mathcal{T}_{-n(r),\nu}\hat{g}_-(\mathcal{T},\bar{z},\sigma) : \qquad (6.1d)$$

$$\hat{A}_+(\sigma) \equiv \langle\hat{g}_+(\mathcal{T}^{(1)},z_1,\sigma)\ldots\hat{g}_+(\mathcal{T}^{(N)},z_N,\sigma)\rangle_\sigma \qquad (6.1e)$$

$$\hat{A}_-(\sigma) \equiv \langle\hat{g}_-(\mathcal{T}^{(1)},\bar{z}_1,\sigma)\ldots\hat{g}_-(\mathcal{T}^{(N)},\bar{z}_N,\sigma)\rangle_\sigma \qquad (6.1f)$$

of the twisted primary fields, and we will generally abbreviate

$$\hat{J}_{n(r)\mu}(z) \equiv \hat{J}_{n(r)\mu}(z,\sigma)\,, \quad \hat{\bar{J}}_{n(r)\mu}(\bar{z}) \equiv \hat{\bar{J}}_{n(r)\mu}(\bar{z},\sigma)\,, \quad \mathcal{T}_{n(r)\mu} \equiv \mathcal{T}_{n(r)\mu}(T,\sigma) \tag{6.2a}$$

$$\hat{g}_{+}(\mathcal{T},z) \equiv \hat{g}_{+}(\mathcal{T},z,\sigma)\,, \quad \hat{g}_{-}(\mathcal{T},\bar{z}) \equiv \hat{g}_{-}(\mathcal{T},\bar{z},\sigma) \tag{6.2b}$$

by suppressing the sector label σ.

We begin with the exact operator products of the currents

$$\hat{J}_{n(r)\mu}(z)\hat{J}_{n(s)\nu}(w) = \frac{\mathcal{G}_{n(r)\mu;n(s)\nu}(\sigma)}{(z-w)^2} + \frac{i\mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(r)+n(s),\delta}(\sigma)\hat{J}_{n(r)+n(s),\delta}(w)}{z-w} + \\ + :\hat{J}_{n(r)\mu}(z)\hat{J}_{n(s)\nu}(w): \tag{6.3a}$$

$$\hat{\bar{J}}_{n(r)\mu}(\bar{z})\hat{\bar{J}}_{n(s)\nu}(\bar{w}) = \frac{\mathcal{G}_{n(r)\mu;n(s)\nu}(\sigma)}{(\bar{z}-\bar{w})^2} + \frac{i\mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(r)+n(s),\delta}(\sigma)\hat{\bar{J}}_{n(r)+n(s),\delta}(\bar{w})}{\bar{z}-\bar{w}} + \\ + :\hat{\bar{J}}_{n(r)\mu}(\bar{z})\hat{\bar{J}}_{n(s)\nu}(\bar{w}): \tag{6.3b}$$

where $:\cdot:$ is operator product normal ordering. Then the mode normal orderings M and $\bar{M}$ are defined as

$$:\hat{J}_{n(r)\mu}\left(m+\frac{n(r)}{\rho(\sigma)}\right)\hat{J}_{n(s)\nu}\left(n+\frac{n(s)}{\rho(\sigma)}\right):_M \equiv \theta\left(m+\frac{n(r)}{\rho(\sigma)}\geq 0\right)\times \\ \times\hat{J}_{n(s)\nu}\left(n+\frac{n(s)}{\rho(\sigma)}\right)\hat{J}_{n(r)\mu}\left(m+\frac{n(r)}{\rho(\sigma)}\right) + \\ +\theta\left(m+\frac{n(r)}{\rho(\sigma)}<0\right)\,\hat{J}_{n(r)\mu}\left(m+\frac{n(r)}{\rho(\sigma)}\right)\hat{J}_{n(s)\nu}\left(n+\frac{n(s)}{\rho(\sigma)}\right) \tag{6.4a}$$

$$:\hat{\bar{J}}_{n(r)\mu}\left(m+\frac{n(r)}{\rho(\sigma)}\right)\hat{\bar{J}}_{n(s)\nu}\left(n+\frac{n(s)}{\rho(\sigma)}\right):_{\bar{M}} \equiv \theta\left(m+\frac{n(r)}{\rho(\sigma)}\leq 0\right)\times \\ \times\hat{\bar{J}}_{n(s)\nu}\left(n+\frac{n(s)}{\rho(\sigma)}\right)\hat{\bar{J}}_{n(r)\mu}\left(m+\frac{n(r)}{\rho(\sigma)}\right) + \\ +\theta(m+\tfrac{n(r)}{\rho(\sigma)}>0)\,\hat{\bar{J}}_{n(r)\mu}\left(m+\frac{n(r)}{\rho(\sigma)}\right)\hat{\bar{J}}_{n(s)\nu}\left(n+\frac{n(s)}{\rho(\sigma)}\right), \tag{6.4b}$$

where M normal ordering was studied for the twisted left movers in ref. [32]. The M and $\bar{M}$ orderings are truly normal only for the semisimple orbifold affine algebras [31]–[35] of the permutation orbifolds (see section 7), but we can also use them to obtain exact relations for any WZW orbifold $A_g(H)/H$.

Using (6.4) and the general twisted current algebra (5.15), we find the exact operator products

$$\hat{J}_{n(r)\mu}(z)\hat{J}_{n(s)\nu}(w) = \left(\frac{w}{z}\right)^{\frac{\bar{n}(r)}{\rho(\sigma)}}\left\{\left[\frac{1}{(z-w)^2}+\frac{\bar{n}(r)/\rho(\sigma)}{w(z-w)}\right]\mathcal{G}_{n(r)\mu;n(s)\nu}(\sigma) + \right. \\ \left. + \frac{i\mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(r)+n(s),\delta}(\sigma)\hat{J}_{n(r)+n(s),\delta}(w)}{z-w}\right\} +$$

$$+ :\hat{J}_{n(r)\mu}(z)\hat{J}_{n(s)\nu}(w):_M \tag{6.5a}$$

$$\begin{aligned}\hat{\bar{J}}_{n(r)\mu}(\bar{z})\hat{\bar{J}}_{n(s)\nu}(\bar{w}) = \left(\frac{\bar{w}}{\bar{z}}\right)^{\frac{\overline{-n(r)}}{\rho(\sigma)}} &\left\{\left[\frac{1}{(\bar{z}-\bar{w})^2} + \frac{\overline{-n(r)}/\rho(\sigma)}{\bar{w}(\bar{z}-\bar{w})}\right]\mathcal{G}_{n(r)\mu;n(s)\nu}(\sigma) + \right.\\ &\left. + \frac{i\mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(r)+n(s),\delta}(\sigma)\hat{\bar{J}}_{n(r)+n(s),\delta}(\bar{w})}{\bar{z}-\bar{w}}\right\} + \\ &+ :\hat{\bar{J}}_{n(r)\mu}(\bar{z})\hat{\bar{J}}_{n(s)\nu}(\bar{w}):_{\bar{M}}\end{aligned} \tag{6.5b}$$

in terms of M and $\bar{M}$ normal ordered products. To obtain these results, we have used the definition of $\bar{n}(r)$ in (3.12e) and the fact that [32]

$$\overline{-n(r)} = \overline{-\bar{n}(r)} = \begin{cases} \rho(\sigma) - \bar{n}(r) & \text{when } \bar{n}(r) \neq 0 \\ 0 & \text{when } \bar{n}(r) = 0 \end{cases} \tag{6.6}$$

which follows from the definition of $\bar{n}(r)$. The left mover result in (6.5a) was given in ref. [32].

Comparing (6.5) with (6.3), we find the relations among the normal ordered products

$$\begin{aligned}&:\hat{J}_{n(r)\mu}(z)\hat{J}_{n(s)\nu}(z):=:\hat{J}_{n(r)\mu}(z)\hat{J}_{n(s)\nu}(z):_M \\ &-\frac{i}{z}\frac{\bar{n}(r)}{\rho(\sigma)}\mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(r)+n(s),\delta}(\sigma)\hat{J}_{n(r)+n(s),\delta}(z) + \frac{1}{z^2}\frac{\bar{n}(r)}{2\rho(\sigma)}\left(1-\frac{\bar{n}(r)}{\rho(\sigma)}\right)\mathcal{G}_{n(r)\mu;n(s)\nu}(\sigma)\end{aligned} \tag{6.7a}$$

$$\begin{aligned}&:\hat{\bar{J}}_{n(r)\mu}(\bar{z})\hat{\bar{J}}_{n(s)\nu}(\bar{z}):=:\hat{\bar{J}}_{n(r)\mu}(\bar{z})\hat{\bar{J}}_{n(s)\nu}(\bar{z}):_{\bar{M}} \\ &-\frac{i}{\bar{z}}\frac{\overline{-n(r)}}{\rho(\sigma)}\mathcal{F}_{n(r)\mu;n(s)\nu}{}^{n(r)+n(s),\delta}(\sigma)\hat{\bar{J}}_{n(r)+n(s),\delta}(\bar{z}) + \frac{1}{\bar{z}^2}\frac{\overline{-n(r)}}{2\rho(\sigma)} \times \\ &\qquad\times\left(1-\frac{\overline{-n(r)}}{\rho(\sigma)}\right)\mathcal{G}_{n(r)\mu;n(s)\nu}(\sigma)\,.\end{aligned} \tag{6.7b}$$

These relations allow us to express the stress tensors of the twisted affine-Sugawara constructions in terms of M and $\bar{M}$ normal ordered products (see appendix A with $\mathcal{L} \to \mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}$). The corresponding Virasoro generators of the twisted affine-Sugawara constructions are

$$L_\sigma(m) = \mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)\mu;-n(r),\nu}(\sigma)\sum_{p\in\mathbb{Z}}:\hat{J}_{n(r)\mu}\left(p+\frac{n(r)}{\rho(\sigma)}\right)\hat{J}_{-n(r),\nu}\left(m-p-\frac{n(r)}{\rho(\sigma)}\right): \tag{6.8a}$$

$$\begin{aligned}= \mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)\mu;-n(r),\nu}(\sigma)&\left\{\sum_{p\in\mathbb{Z}}:\hat{J}_{n(r)\mu}\left(p+\frac{n(r)}{\rho(\sigma)}\right)\hat{J}_{-n(r),\nu}\left(m-p-\frac{n(r)}{\rho(\sigma)}\right):_M - \right.\\ &- i\frac{\bar{n}(r)}{\rho(\sigma)}\mathcal{F}_{n(r)\mu;-n(r),\nu}{}^{0\delta}(\sigma)\hat{J}_{0\delta}(m) + \\ &\left. + \delta_{m,0}\frac{\bar{n}(r)}{2\rho(\sigma)}\left(1-\frac{\bar{n}(r)}{\rho(\sigma)}\right)\mathcal{G}_{n(r)\mu;-n(r),\nu}(\sigma)\right\}\end{aligned} \tag{6.8b}$$

$$\bar{L}_\sigma(m) = \mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)\mu;-n(r),\nu}(\sigma) \sum_{p\in\mathbb{Z}} : \hat{\bar{J}}_{n(r)\mu}\left(p+\frac{n(r)}{\rho(\sigma)}\right) \hat{\bar{J}}_{-n(r),\nu}\left(-m-p-\frac{n(r)}{\rho(\sigma)}\right) : \tag{6.8c}$$

$$\begin{aligned} &= \mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)\mu;-n(r),\nu}(\sigma) \Bigg\{ \sum_{p\in\mathbb{Z}} : \hat{\bar{J}}_{n(r)\mu}\left(p+\frac{n(r)}{\rho(\sigma)}\right) \hat{\bar{J}}_{-n(r),\nu}\left(-m-p-\frac{n(r)}{\rho(\sigma)}\right) :_{\bar{M}} - \\ &\qquad - i\overline{\frac{-n(r)}{\rho(\sigma)}} \mathcal{F}_{n(r)\mu;-n(r),\nu}{}^{0\delta}(\sigma) \hat{\bar{J}}_{0\delta}(-m) + \\ &\qquad + \delta_{m,0} \frac{\overline{-n(r)}}{2\rho(\sigma)} \left(1-\overline{\frac{-n(r)}{\rho(\sigma)}}\right) \mathcal{G}_{n(r)\mu;-n(r),\nu}(\sigma) \Bigg\}. \end{aligned} \tag{6.8d}$$

$$\frac{\overline{-n(r)}}{2\rho(\sigma)} \left(1-\overline{\frac{-n(r)}{\rho(\sigma)}}\right) = \frac{\bar{n}(r)}{2\rho(\sigma)} \left(1-\frac{\bar{n}(r)}{\rho(\sigma)}\right) \tag{6.8e}$$

in sector σ of each WZW orbifold $A_g(H)/H$. The identity in (6.8e) follows from (6.6).

We turn next to the exact operator products of the twisted currents with the twisted primary fields, starting with the operator products

$$\hat{J}_{n(r)\mu}(z)\hat{g}_+(\mathcal{T},w) = \frac{1}{z-w}\hat{g}_+(\mathcal{T},w)\mathcal{T}_{n(r)\mu} + : \hat{J}_{n(r)\mu}(z)\hat{g}_+(\mathcal{T},w) : \tag{6.9a}$$

$$\hat{\bar{J}}_{n(r)\mu}(\bar{z})\hat{g}_-(\mathcal{T},\bar{w}) = -\frac{1}{\bar{z}-\bar{w}}\mathcal{T}_{n(r)\mu}\hat{g}_-(\mathcal{T},\bar{w}) + : \hat{\bar{J}}_{n(r)\mu}(\bar{z})\hat{g}_-(\mathcal{T},\bar{w}) : \tag{6.9b}$$

in terms of operator product normal ordering. For these operators, we introduce the corresponding M and $\bar{M}$ normal ordering as follows

$$\begin{aligned} &: \hat{J}_{n(r)\mu}\left(m+\frac{n(r)}{\rho(\sigma)}\right) \hat{g}_+(\mathcal{T},z) :_M \equiv \\ &\quad \equiv \theta\left(m+\frac{n(r)}{\rho(\sigma)} \geq 0\right) \hat{g}_+(\mathcal{T},z)\hat{J}_{n(r)\mu}\left(m+\frac{n(r)}{\rho(\sigma)}\right) + \theta\left(m+\frac{n(r)}{\rho(\sigma)} < 0\right) \times \\ &\qquad \times \hat{J}_{n(r)\mu}\left(m+\frac{n(r)}{\rho(\sigma)}\right) \hat{g}_+(\mathcal{T},z) \end{aligned} \tag{6.10a}$$

$$\begin{aligned} &: \hat{\bar{J}}_{n(r)\mu}\left(m+\frac{n(r)}{\rho(\sigma)}\right) \hat{g}_-(\mathcal{T},\bar{z}) :_{\bar{M}} \equiv \\ &\quad \equiv \theta\left(m+\frac{n(r)}{\rho(\sigma)} \leq 0\right) \hat{g}_-(\mathcal{T},\bar{z})\hat{\bar{J}}_{n(r)\mu}\left(m+\frac{n(r)}{\rho(\sigma)}\right) + \theta\left(m+\frac{n(r)}{\rho(\sigma)} > 0\right) \times \\ &\qquad \times \hat{\bar{J}}_{n(r)\mu}\left(m+\frac{n(r)}{\rho(\sigma)}\right) \hat{g}_-(\mathcal{T},\bar{z}) . \end{aligned} \tag{6.10b}$$

Again, these mode orderings are truly normal only for permutation orbifolds (see section 7). Then using the commutators in (5.20a), (5.20b) we find the exact operator products

$$\hat{J}_{n(r)\mu}(z)\hat{g}_+(\mathcal{T},w) = \left(\frac{w}{z}\right)^{\frac{\bar{n}(r)}{\rho(\sigma)}} \frac{1}{z-w}\hat{g}_+(\mathcal{T},w)\mathcal{T}_{n(r)\mu} + : \hat{J}_{n(r)\mu}(z)\hat{g}_+(\mathcal{T},w) :_M \tag{6.11a}$$

$$\hat{\bar{J}}_{n(r)\mu}(\bar{z})\hat{g}_-(\mathcal{T},\bar{w}) = -\left(\frac{\bar{w}}{\bar{z}}\right)^{\frac{\overline{-n(r)}}{\rho(\sigma)}} \frac{1}{\bar{z}-\bar{w}}\mathcal{T}_{n(r)\mu}\hat{g}_-(\mathcal{T},\bar{w}) + :\hat{\bar{J}}_{n(r)\mu}(\bar{z})\hat{g}_-(\mathcal{T},\bar{w}):_{\bar{M}} \tag{6.11b}$$

in terms of the M and $\bar{M}$ normal ordered products.

Comparing (6.11) with (6.9), we also obtain the relations among the normal ordered products

$$\begin{aligned} :\hat{J}_{n(r)\mu}(z)\hat{g}_+(\mathcal{T},z): &= :\hat{J}_{n(r)\mu}(z)\hat{g}_+(\mathcal{T},z):_M - \frac{\bar{n}(r)}{\rho(\sigma)}\frac{1}{z}\hat{g}_+(\mathcal{T},z)\mathcal{T}_{n(r)\mu} \\ :\hat{\bar{J}}_{n(r)\mu}(\bar{z})\hat{g}_-(\mathcal{T},\bar{z}): &= :\hat{\bar{J}}_{n(r)\mu}(\bar{z})\hat{g}_-(\mathcal{T},\bar{z}):_{\bar{M}} + \frac{\overline{-n(r)}}{\rho(\sigma)}\frac{1}{\bar{z}}\mathcal{T}_{n(r)\mu}\hat{g}_-(\mathcal{T},\bar{z}) \end{aligned} \tag{6.12}$$

in each sector σ of each $A_g(H)/H$.

6.2 Mode-ordered form of the vertex operator equations

As an application of (6.12), we may reexpress the twisted vertex operator equations (6.1c), (6.1d)

$$\begin{aligned} \partial\hat{g}_+(\mathcal{T},z) &= 2\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)\mu;-n(r),\nu}(\sigma) \times \qquad (6.13a) \\ &\quad \times \left(:\hat{J}_{n(r)\mu}(z)\hat{g}_+(\mathcal{T},z):_M - \frac{\bar{n}(r)}{\rho(\sigma)}\frac{1}{z}\hat{g}_+(\mathcal{T},z)\mathcal{T}_{n(r)\mu}\right)\mathcal{T}_{-n(r),\nu} \\ \bar{\partial}\hat{g}_-(\mathcal{T},\bar{z}) &= -2\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)\mu;-n(r),\nu}(\sigma) \times \qquad (6.13b) \\ &\quad \times\mathcal{T}_{-n(r),\nu}\left(:\hat{\bar{J}}_{n(r)\mu}(\bar{z})\hat{g}_-(\mathcal{T},\bar{z}):_{\bar{M}} + \frac{\overline{-n(r)}}{\rho(\sigma)}\frac{1}{\bar{z}}\mathcal{T}_{n(r)\mu}\hat{g}_-(\mathcal{T},\bar{z})\right) \end{aligned}$$

in terms of M and $\bar{M}$ normal ordered products.

Using eq. (6.10), we give the explicit form of these mode ordered products

$$:\hat{J}_{n(r)\mu}(z)\hat{g}_+(\mathcal{T},z):_M = \hat{J}^-_{n(r)\mu}(z)\hat{g}_+(\mathcal{T},z) + \hat{g}_+(\mathcal{T},z)\hat{J}^+_{n(r)\mu}(z) \tag{6.14a}$$

$$\begin{aligned} \hat{J}^-_{n(r)\mu}(z) &\equiv \sum_m \theta\left(m+\frac{n(r)}{\rho(\sigma)}<0\right)\hat{J}_{n(r)\mu}\left(m+\frac{n(r)}{\rho(\sigma)}\right)z^{-\left(m+\frac{n(r)}{\rho(\sigma)}\right)-1} && (6.14b) \\ &= \sum_{m\le -1}\hat{J}_{n(r)\mu}\left(m+\frac{\bar{n}(r)}{\rho(\sigma)}\right)z^{-\left(m+\frac{\bar{n}(r)}{\rho(\sigma)}\right)-1} && (6.14c) \\ \hat{J}^+_{n(r)\mu}(z) &\equiv \sum_m \theta\left(m+\frac{n(r)}{\rho(\sigma)}\ge 0\right)\hat{J}_{n(r)\mu}\left(m+\frac{n(r)}{\rho(\sigma)}\right)z^{-(m+\frac{n(r)}{\rho(\sigma)})-1} && (6.14d) \\ &= \sum_{m\ge 0}\hat{J}_{n(r)\mu}\left(m+\frac{\bar{n}(r)}{\rho(\sigma)}\right)z^{-\left(m+\frac{\bar{n}(r)}{\rho(\sigma)}\right)-1} && (6.14e) \end{aligned}$$

$$: \hat{\bar{J}}_{n(r)\mu}(\bar{z})\hat{g}_-(\mathcal{T},\bar{z}) :_{\bar{M}} = \hat{\bar{J}}^{+}_{n(r)\mu}(\bar{z})\hat{g}_-(\mathcal{T},\bar{z}) + \hat{g}_-(\mathcal{T},\bar{z})\hat{\bar{J}}^{-}_{n(r)\mu}(\bar{z}) \tag{6.15a}$$

$$\hat{\bar{J}}^{+}_{n(r)\mu}(\bar{z}) \equiv \sum_m \theta\left(m+\frac{n(r)}{\rho(\sigma)}>0\right)\hat{\bar{J}}_{n(r)\mu}\left(m+\frac{n(r)}{\rho(\sigma)}\right)\bar{z}^{\left(m+\frac{n(r)}{\rho(\sigma)}\right)-1} \tag{6.15b}$$

$$= \sum_{m\geq 0}\hat{\bar{J}}_{n(r)\mu}\left(m+\frac{\bar{n}(r)}{\rho(\sigma)}\right)\bar{z}^{\left(m+\frac{\bar{n}(r)}{\rho(\sigma)}\right)-1} - \delta_{\bar{n}(r),0}\hat{\bar{J}}_{0\mu}(0)\bar{z}^{-1} \tag{6.15c}$$

$$= \sum_{m>0}\hat{\bar{J}}_{n(r)\mu}\left(m-\frac{\overline{-n(r)}}{\rho(\sigma)}\right)\bar{z}^{\left(m-\frac{\overline{-n(r)}}{\rho(\sigma)}\right)-1} \tag{6.15d}$$

$$\hat{\bar{J}}^{-}_{n(r)\mu}(\bar{z}) \equiv \sum_m \theta\left(m+\frac{n(r)}{\rho(\sigma)}\leq 0\right)\hat{\bar{J}}_{n(r)\mu}\left(m+\frac{n(r)}{\rho(\sigma)}\right)\bar{z}^{\left(m+\frac{n(r)}{\rho(\sigma)}\right)-1} \tag{6.15e}$$

$$= \sum_{m<0}\hat{\bar{J}}_{n(r)\mu}\left(m+\frac{\bar{n}(r)}{\rho(\sigma)}\right)\bar{z}^{\left(m+\frac{\bar{n}(r)}{\rho(\sigma)}\right)-1} + \delta_{\bar{n}(r),0}\hat{\bar{J}}_{0\mu}(0)\bar{z}^{-1} \tag{6.15f}$$

$$= \sum_{m\leq 0}\hat{\bar{J}}_{n(r)\mu}\left(m-\frac{\overline{-n(r)}}{\rho(\sigma)}\right)\bar{z}^{\left(m-\frac{\overline{-n(r)}}{\rho(\sigma)}\right)-1}, \tag{6.15g}$$

where $\hat{J}^{\pm}$, $\hat{\bar{J}}^{\pm}$ are called the twisted partial currents. In obtaining the final form for the twisted right mover partial currents, we also used the identity

$$\hat{\bar{J}}_{n(r)\mu}\left(m-\frac{\overline{-n(r)}}{\rho(\sigma)}\right) = \hat{\bar{J}}_{n(r)\mu}\left(m-1+\frac{\bar{n}(r)}{\rho(\sigma)}\right), \qquad \bar{n}(r)\neq 0 \tag{6.16}$$

which follows from (6.6).

Finally, it will be useful to know the commutation relations of the partial currents

$$[\hat{J}^{+}_{n(r)\mu}(z),\hat{g}_+(\mathcal{T},w)] = \frac{1}{z-w}\left(\frac{w}{z}\right)^{\frac{\bar{n}(r)}{\rho(\sigma)}}\hat{g}_+(\mathcal{T},z)\mathcal{T}_{n(r)\mu}, \qquad |z|>|w| \tag{6.17a}$$

$$[\hat{J}^{-}_{n(r)\mu}(z),\hat{g}_+(\mathcal{T},w)] = -\frac{1}{z-w}\left(\frac{w}{z}\right)^{\frac{\bar{n}(r)}{\rho(\sigma)}}\hat{g}_+(\mathcal{T},z)\mathcal{T}_{n(r)\mu}, \qquad |w|>|z| \tag{6.17b}$$

and similar relations for the twisted right movers.

6.3 A consistency check on the twisted vertex operator equations

The forms (5.13) (and hence the forms (6.13)) of the twisted vertex operator equations followed from local isomorphisms. In this subsection, we provide a consistency check on these results, starting with the differential equations

$$\partial\hat{g}_+(\mathcal{T},z) = [L_\sigma(-1),\hat{g}_+(\mathcal{T},z)], \qquad \bar{\partial}\hat{g}_-(\mathcal{T},\bar{z}) = [\bar{L}_\sigma(-1),\hat{g}_-(\mathcal{T},\bar{z})] \tag{6.18}$$

which are special cases of the more general commutation relations in (5.20c), (5.20d).

Then using (6.8a), (6.8b) and (5.20a), (5.20b) one finds after some algebra a somewhat different form of the twisted vertex operator equations

$$\begin{aligned}\partial\hat{g}_+(\mathcal{T},z) = \mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)\mu;-n(r),\nu}(\sigma)\Bigg(&2:\hat{J}_{n(r)\mu}(z)\hat{g}_+(\mathcal{T},z):_M \mathcal{T}_{-n(r),\nu} - \\ &- \hat{g}_+(\mathcal{T},z)\frac{1}{z}\mathcal{T}_{n(r)\mu}\mathcal{T}_{-n(r),\nu}(1-\delta_{\bar{n}(r),0}) - \\ &- i\frac{\bar{n}(r)}{\rho(\sigma)}\frac{1}{z}\hat{g}_+(\mathcal{T},z)\mathcal{F}_{n(r)\mu;-n(r),\nu}{}^{0,\delta}\mathcal{T}_{0,\delta}\Bigg) \end{aligned} \tag{6.19a}$$

$$\begin{aligned}\bar{\partial}\hat{g}_-(\mathcal{T},\bar{z}) = -\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)\mu;-n(r),\nu}(\sigma)\Bigg(&2\mathcal{T}_{-n(r),\nu}:\hat{\bar{J}}_{n(r)\mu}(\bar{z})\hat{g}_-(\mathcal{T},\bar{z}):_{\bar{M}} + \\ &+ \frac{1}{\bar{z}}\mathcal{T}_{-n(r),\nu}\mathcal{T}_{n(r)\mu}\hat{g}_-(\mathcal{T},\bar{z})(1-\delta_{\bar{n}(r),0}) - \\ &- \frac{i}{\bar{z}}\frac{\overline{-n(r)}}{\rho(\sigma)}\mathcal{F}_{n(r)\mu;-n(r),\nu}{}^{0,\delta}\mathcal{T}_{0,\delta}\hat{g}_-(\mathcal{T},\bar{z})\Bigg)\,. \end{aligned} \tag{6.19b}$$

These forms agree with those given in (6.13) iff

$$\begin{aligned}&2\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)\mu;-n(r),\nu}(\sigma)\frac{\bar{n}(r)}{\rho(\sigma)}\mathcal{T}_{n(r)\mu}\mathcal{T}_{-n(r),\nu} = \\ &\quad = \mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)\mu;-n(r),\nu}(\sigma)\left(\mathcal{T}_{n(r)\mu}\mathcal{T}_{-n(r),\nu}(1-\delta_{\bar{n}(r),0}) + i\frac{\bar{n}(r)}{\rho(\sigma)}\mathcal{F}_{n(r)\mu;-n(r),\nu}{}^{0,\delta}\mathcal{T}_{0,\delta}\right)\end{aligned} \tag{6.20a}$$

$$\begin{aligned}&2\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)\mu;-n(r),\nu}(\sigma)\frac{\overline{-n(r)}}{\rho(\sigma)}\mathcal{T}_{-n(r),\nu}\mathcal{T}_{n(r)\mu} = \\ &\quad = \mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)\mu;-n(r),\nu}(\sigma)\left(\mathcal{T}_{-n(r),\nu}\mathcal{T}_{n(r)\mu}(1-\delta_{\bar{n}(r),0}) - i\frac{\overline{-n(r)}}{\rho(\sigma)}\mathcal{F}_{n(r)\mu;-n(r),\nu}{}^{0,\delta}\mathcal{T}_{0,\delta}\right).\end{aligned} \tag{6.20b}$$

These consistency relations are in fact identities. To check the first identity, replace the $i\mathcal{F}_{n(r)\mu;-n(r),\nu}{}^{0,\delta}\mathcal{T}_{0,\delta}$ term by the commutator $[\mathcal{T}_{n(r)\mu},\mathcal{T}_{-n(r),\nu}]$ and note that the identity is trivially satisfied for the $\bar{n}(r)=0$ terms. For the $\bar{n}\neq 0$ terms, use the symmetry (4.24b) of $\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}$ and the change of dummy variables $n\to -n$, which induces $\bar{n}/\rho \to 1-\bar{n}/\rho$ (see eq. (6.6)). The second identity then follows from the first identity by $n(r)\to -n(r)$ and the symmetry (4.24b) of $\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}$.

7. The WZW permutation orbifolds

7.1 Systematics

In this and the following two sections, we apply our general development above to the special case of the WZW permutation orbifolds

$$\frac{A_g(H)}{H}, \qquad H(\text{permutation}) \subset S_N\ . \tag{7.1}$$

We begin by reprising the salient features of the permutation-invariant theory $A_g(H)$

$$g = \oplus_{I=0}^{K-1}\mathfrak{g}^I, \quad K \le N\,, \quad \mathfrak{g}^I \simeq \mathfrak{g}\,, \quad a \longrightarrow aI, \quad \alpha \longrightarrow \alpha I\,, \tag{7.2a}$$
$$G_{aI,bJ} = k\eta_{ab}\delta_{IJ}\,, \quad f_{aI,bJ}{}^{cK} = f_{ab}{}^c\delta_{IJ}\delta_J{}^K\,, \quad (T_{aI})_{\alpha J}{}^{\beta K} = (T_a)_\alpha{}^\beta\delta_{IJ}\delta_J{}^K \tag{7.2b}$$
$$I = 0\,,\ldots,K-1\,, \quad a = 1\ldots\dim\mathfrak{g}, \quad \alpha = 1\ldots\dim T \tag{7.2c}$$

(see also the relations in (2.9)) where I is the semisimplicity index. Here each copy $\mathfrak{g}^I$ is taken at level k with Killing metric η_{ab}, structure constants $f_{ab}{}^c$ and irrep T_a.

We next discuss the form of the symmetry transformations w and W when H is any subgroup of S_N. The action in the adjoint is [32]

$$w(h_\sigma)_{aI}{}^{bJ} = \delta_a{}^b w(h_\sigma)_I{}^J\,, \quad \forall\ h_\sigma \in H(\text{permutation}) \subset \text{Aut}(g)\,. \tag{7.3}$$

Then we must find $W(h_\sigma;T)$, the corresponding action of h_σ in matrix representation T. It is straightforward to check that

$$W(h_\sigma;T)_{\alpha I}{}^{\beta J} = \delta_\alpha{}^\beta w(h_\sigma)_I{}^J, \qquad \forall\ h_\sigma(\text{permutation}) \subset \text{Aut}(g) \tag{7.4}$$

solves the linkage relation (3.5a) so long as

$$w^\dagger(h_\sigma)_I{}^J w(h_\sigma)_J{}^K = w(h_\sigma)_J{}^I\delta_I{}^K\,, \quad \forall\ I,J,K \tag{7.5}$$

and these relations indeed hold for all $h_\sigma \in H$ because each $w(h_\sigma)_I{}^J$ is a permutation matrix.

The simple forms (7.3) and (7.4) tell us that, for permutation orbifolds, the extended H-eigenvalue problem (3.13) and the H-eigenvalue problem (3.12) are both simply related to the solution of the reduced eigenvalue problem for $w(h_\sigma)_I{}^J$

$$\begin{aligned}
&n(r)\mu \longrightarrow n(r)aj, \quad N(r)\mu \longrightarrow N(r)\alpha j\\
&R(T,\sigma) = \rho(\sigma), \quad E(T,\sigma) = E(\sigma), \quad N(r) = n(r)\\
&U^\dagger(T,\sigma)_{\alpha I}{}^{n(r)\beta j} = \delta_\alpha{}^\beta U^\dagger(\sigma)_I{}^{n(r)j}, \quad U^\dagger(\sigma)_{aI}{}^{n(r)bj} = \delta_a{}^b U^\dagger(\sigma)_I^{n(r)j}\\
&w(h_\sigma)_I{}^J U^\dagger(\sigma)_J{}^{n(r)j} = U^\dagger(\sigma)_I{}^{n(r)j}E_{n(r)}(\sigma), \quad E_{n(r)}(\sigma) = e^{-2\pi i\frac{n(r)}{\rho(\sigma)}}
\end{aligned} \tag{7.6}$$

which is given as the last equation in (7.6). This gives the simplified forms of the duality transformations

$$\mathcal{G}_{n(r)aj;n(s)bl}(\sigma) = k\eta_{ab}\chi(\sigma)_{n(r)aj}\chi(\sigma)_{n(s)bl}\sum_I U(\sigma)_{n(r)j}{}^I U(\sigma)_{n(s)l}{}^I \tag{7.7a}$$

$$\mathcal{G}^{n(r)aj;n(s)bl}(\sigma) = \frac{\eta^{ab}}{k}\chi(\sigma)^{-1}_{n(r)aj}\chi(\sigma)^{-1}_{n(s)bl}\sum_I U(\sigma)_I{}^{n(r)j}U(\sigma)_I{}^{n(s)l} \tag{7.7b}$$

$$\mathcal{F}_{n(r)aj;n(s)bl}{}^{n(t)cm}(\sigma) = f_{ab}{}^c\chi(\sigma)_{n(r)aj}\chi(\sigma)_{n(s)bl}\chi(\sigma)^{-1}_{n(t)cm}\times$$

$$\times \sum_I U(\sigma)_{n(r)j}{}^I U(\sigma)_{n(s)l}{}^I U^\dagger(\sigma)_I{}^{n(t)m} \tag{7.7c}$$

$$\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)aj;n(s)bl}(\sigma) = \frac{k}{2k+Q_{\mathfrak{g}}}\mathcal{G}^{n(r)aj;n(s)bl}(\sigma) \tag{7.7d}$$

$$(\mathcal{T}_{n(r)aj})_{n(s)\alpha l}{}^{n(t)\beta m} = (T_a)_\alpha{}^\beta \chi(\sigma)_{n(r)aj} \sum_I U(\sigma)_{n(r)j}{}^I U(\sigma)_{n(s)l}{}^I U^\dagger(\sigma)_I{}^{n(t)m} \tag{7.7e}$$

$$\mathcal{M}(\mathcal{T},\sigma)_{n(r)\alpha j}{}^{n(s)\beta l} = \frac{k}{y(T)}(\mathbb{1})_{n(r)\alpha j}{}^{n(s)\beta l}$$

$$(\mathbb{1})_{n(r)\alpha j}{}^{n(s)\beta l} = \delta_{n(r)-n(s),0 \bmod \rho(\sigma)}\delta_\alpha^\beta \delta_j^l \tag{7.7f}$$

$$\mathcal{D}_{\hat{\mathfrak{g}}(\sigma)}(\mathcal{T}) = \mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)aj;-n(r),bl}(\sigma)\mathcal{T}_{n(r)aj}\mathcal{T}_{-n(r),bl} = \Delta_{\mathfrak{g}}(T)\mathbb{1} \tag{7.7g}$$

where $\Delta_{\mathfrak{g}}(T)$ is the conformal weight of irrep T under any copy $\mathfrak{g}^I \simeq \mathfrak{g}$. More explicit forms of $\mathcal{G}$, $\mathcal{F}$, $\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}$ and $\mathcal{T}$ are given in subsections 7.3, 7.4, 7.5.

Because the twisted conformal weight matrix (7.7g) is proportional to unity, we also find the simpler OPEs

$$\hat{g}(\mathcal{T},\bar{z},z)_{n(r)\alpha j}{}^{n(s)\beta l} = (\hat{g}_-(\mathcal{T},z)\hat{g}_+(\mathcal{T},\bar{z}))_{n(r)\alpha j}{}^{n(s)\beta l}, \qquad \alpha,\beta = 1\ldots \dim T \tag{7.8a}$$

$$\hat{T}_\sigma(z)\hat{g}_+(\mathcal{T},w) = \left(\frac{\Delta_{\mathfrak{g}}(T)}{(z-w)^2} + \frac{\partial_w}{z-w}\right)\hat{g}_+(\mathcal{T},w) + \mathcal{O}(z-w)^0 \tag{7.8b}$$

and $\hat{\bar{T}}_\sigma\ \hat{g}_-$ is a barred copy of (7.8b).

7.2 The ground state of sector σ and simple results

For the permutation orbifolds, the ground state (twist field state) of sector σ satisfies

$$0 = \hat{J}_{n(r)aj}\left(m + \frac{n(r)}{\rho(\sigma)} \geq 0\right)|0\rangle_\sigma = {}_\sigma\langle 0|\hat{J}_{n(r)aj}\left(m+\frac{n(r)}{\rho(\sigma)} \leq 0\right) \tag{7.9a}$$

$$0 = \hat{\bar{J}}_{n(r)aj}\left(m + \frac{n(r)}{\rho(\sigma)} \leq 0\right)|0\rangle_\sigma = {}_\sigma\langle 0|\hat{\bar{J}}_{n(r)aj}\left(m+\frac{n(r)}{\rho(\sigma)} \geq 0\right) \tag{7.9b}$$

$$\begin{aligned}\langle \hat{J}_{n(r)aj}(z)\rangle_\sigma &= \langle \hat{\bar{J}}_{n(r)aj}(\bar{z})\rangle_\sigma \\ &= \langle :\hat{J}_{n(r)aj}(z)\hat{J}_{n(s)bl}(w):_M\rangle_\sigma \\ &= \langle :\hat{\bar{J}}_{n(r)aj}(\bar{z})\hat{\bar{J}}_{n(s)bl}(\bar{w}):_{\bar{M}}\rangle_\sigma = 0\,.\end{aligned} \tag{7.9c}$$

The left mover conditions (7.9a) were given in ref. [32] and the right mover conditions (7.9b) can be understood (see also subsection 7.5) from the mode-number reversed form (5.22) of the twisted right mover current algebra.

Using the exact operator products in (6.5) we find that the current-current correlators

$$\langle \hat{J}_{n(r)aj}(z)\hat{J}_{n(s)bl}(w)\rangle_\sigma = \delta_{n(r)+n(s),0 \bmod \rho(\sigma)}\left(\frac{w}{z}\right)^{\frac{\bar{n}(r)}{\rho(\sigma)}} \times$$

$$
\times \left[\frac{1}{(z-w)^2} + \frac{\bar{n}(r)/\rho(\sigma)}{w(z-w)} \right] \mathcal{G}_{n(r)aj;-n(r),bl}(\sigma) \tag{7.10a}
$$

$$
\begin{aligned}
\langle \hat{\bar{J}}_{n(r)aj}(\bar{z}) \hat{\bar{J}}_{n(s)bl}(\bar{w}) \rangle_\sigma &= \delta_{n(r)+n(s),0 \,\mathrm{mod}\rho(\sigma)} \left(\frac{\bar{w}}{\bar{z}} \right)^{\frac{\overline{-n(r)}}{\rho(\sigma)}} \times \\
&\quad \times \left[\frac{1}{(\bar{z}-\bar{w})^2} + \frac{\overline{-n(r)}/\rho(\sigma)}{\bar{w}(\bar{z}-\bar{w})} \right] \mathcal{G}_{n(r)aj;-n(r),bl}(\sigma)
\end{aligned} \tag{7.10b}
$$

follow for all permutation orbifolds from the ground state conditions (7.9). The left mover result (7.10a) was given in ref. [32].

Using (7.7b), (7.7c), (7.7d) and (6.8), we find that there are no $\hat{J}$ or $\hat{\bar{J}}$ linear terms in the Virasoro generators of the WZW permutation orbifolds

$$
\begin{aligned}
L_\sigma(m) &= \mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)aj;-n(r),bl}(\sigma) \left\{ \sum_{p\in\mathbb{Z}} : \hat{J}_{n(r)aj}\left(p + \frac{n(r)}{\rho(\sigma)}\right) \hat{J}_{-n(r),bl}\left(m - p - \frac{n(r)}{\rho(\sigma)}\right) :_M + \right. \\
&\quad \left. + \delta_{m,0} \frac{\bar{n}(r)}{2\rho(\sigma)} \left(1 - \frac{\bar{n}(r)}{\rho(\sigma)}\right) \mathcal{G}_{n(r)aj;-n(r),bl}(\sigma) \right\}
\end{aligned} \tag{7.11a}
$$

$$
\begin{aligned}
\bar{L}_\sigma(m) &= \mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)aj;-n(r),bl}(\sigma) \left\{ \sum_{p\in\mathbb{Z}} : \hat{\bar{J}}_{n(r)aj}\left(p + \frac{n(r)}{\rho(\sigma)}\right) \hat{\bar{J}}_{-n(r),bl}\left(-m - p - \frac{n(r)}{\rho(\sigma)}\right) :_{\bar{M}} + \right. \\
&\quad \left. + \delta_{m,0} \frac{\overline{-n(r)}}{2\rho(\sigma)} \left(1 - \frac{\overline{-n(r)}}{\rho(\sigma)}\right) \mathcal{G}_{n(r)aj;-n(r),bl}(\sigma) \right\} .
\end{aligned} \tag{7.11b}
$$

Then the ground state conditions (7.9a), (7.9b) allow us to compute the ground state conformal weights (conformal weights of the twist fields)

$$
L_\sigma(m \geq 0)|0\rangle_\sigma = \delta_{m,0}\hat{\Delta}_0(\sigma)|0\rangle_\sigma \,, \qquad \bar{L}_\sigma(m \geq 0)|0\rangle_\sigma = \delta_{m,0}\hat{\bar{\Delta}}_0(\sigma)|0\rangle_\sigma \tag{7.12a}
$$

$$
\begin{aligned}
\hat{\Delta}_0(\sigma) &= \frac{c_{\mathfrak{g}}}{2} \sum_r \frac{\bar{n}(r)}{2\rho(\sigma)} \left(1 - \frac{\bar{n}(r)}{\rho(\sigma)}\right) \dim[\bar{n}(r)] \\
&= \frac{c_{\mathfrak{g}}}{2} \sum_r \frac{\overline{-n(r)}}{2\rho(\sigma)} \left(1 - \frac{\overline{-n(r)}}{\rho(\sigma)}\right) \dim[\bar{n}(r)] = \hat{\bar{\Delta}}_0(\sigma) \,,
\end{aligned} \tag{7.12b}
$$

$$
c_{\mathfrak{g}} = \frac{2k \dim \mathfrak{g}}{2k + Q_{\mathfrak{g}}} \,. \tag{7.12c}
$$

The left mover result $\hat{\Delta}_0(\sigma)$ was given in ref. [32]. Here $\dim[\bar{n}(r)]$ is the dimension of the degenerate subspace labeled by the spectral index $\bar{n}(r)$. The equality of the left and right mover ground state conformal weights follows from (6.8e), and this equality provides a consistency check on the form (5.15) of the general twisted current algebra. For the cyclic permutation orbifolds and the S_N permutation orbifolds, more explicit forms of these conformal weights are given in ref. [32]. Similarly, one finds equality of the ground state conformal weights in each sector of the general permutation orbifold (see appendix A).

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7.3 The cyclic permutation orbifolds

In this and the following subsection we discuss the cyclic permutation orbibolds $A_g(\mathbb{Z}_\lambda)/\mathbb{Z}_\lambda$ and the S_N permutation orbifolds $A_g(S_N)/S_N$ in further detail, concentrating on the *rectification problem* and the twisted representation matrices for these cases.

For the cyclic permutation group $H = \mathbb{Z}_\lambda$, it is known [31, 32] that

$$n(r) = r\,, \quad \bar{n}(r) = \bar{r}\,, \quad E_r(\sigma) = e^{-2\pi i \frac{r}{\rho(\sigma)}} \tag{7.13a}$$

$$U^\dagger(\sigma)_I{}^{rj} = \frac{1}{\sqrt{\rho(\sigma)}} e^{2\pi i \frac{N(\sigma)r(j-I)}{\lambda}} \delta_{j, I \bmod \frac{\lambda}{\rho(\sigma)}}, \quad \chi(\sigma)_{raj} = \sqrt{\rho(\sigma)} \tag{7.13b}$$

$$\begin{aligned} \mathcal{G}_{raj;sbl}(\sigma) &= \rho(\sigma) k \eta_{ab} \delta_{jl} \delta_{r+s,0 \bmod \rho(\sigma)}, \\ \mathcal{G}^{raj;sbl}(\sigma) &= \frac{1}{\rho(\sigma)k} \eta^{ab} \delta^{jl} \delta_{r+s,0 \bmod \rho(\sigma)}, \end{aligned} \tag{7.13c}$$

$$\begin{aligned} \mathcal{F}_{raj;sbl}{}^{tcm}(\sigma) &= f_{ab}{}^c \delta_{jl} \delta_l{}^m \delta_{r+s-t,0 \bmod \rho(\sigma)} \\ \mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{raj;sbl}(\sigma) &= \frac{k}{2k+Q_{\mathfrak{g}}} \mathcal{G}^{raj;sbl}(\sigma) \end{aligned} \tag{7.13d}$$

$$\begin{aligned} &a = 1 \ldots \dim \mathfrak{g}, \quad \bar{r} = 0, \ldots, \rho(\sigma) - 1, \\ &j = 0, \ldots, \frac{\lambda}{\rho(\sigma)} - 1, \quad \sigma = 0, \ldots, \lambda - 1 \end{aligned} \tag{7.13e}$$

where $I = 0, \ldots, \lambda - 1$ and the integers $N(\sigma)$ are defined in ref. [31]. In the standard notation $\hat{J}^{(r)}_{aj} \equiv \hat{J}_{raj}$, $\hat{\bar{J}}^{(r)}_{aj} \equiv \hat{\bar{J}}_{raj}$, this gives the twisted current algebra of the cyclic permutation orbifolds:

$$\left[\hat{J}^{(r)}_{aj}\left(m + \frac{r}{\rho(\sigma)}\right), \hat{J}^{(s)}_{bl}\left(n + \frac{s}{\rho(\sigma)}\right) \right] = \tag{7.14a}$$

$$= \delta_{jl}\left(i f_{ab}{}^c \hat{J}^{(r+s)}_{cj}\left(m + n + \frac{r+s}{\rho(\sigma)}\right) + \rho(\sigma) k \eta_{ab}\left(m + \frac{r}{\rho(\sigma)}\right) \delta_{m+n+\frac{r+s}{\rho(\sigma)},0} \right)$$

$$\left[\hat{\bar{J}}^{\sharp}{}_{aj}^{(r)}\left(m + \frac{r}{\rho(\sigma)}\right), \hat{\bar{J}}^{\sharp}{}_{bl}^{(s)}\left(n + \frac{s}{\rho(\sigma)}\right) \right] = \tag{7.14b}$$

$$= \delta_{jl}\left(i f_{ab}{}^c \hat{\bar{J}}^{\sharp}{}_{cj}^{(r+s)}\left(m + n + \frac{r+s}{\rho(\sigma)}\right) + +\rho(\sigma) k \eta_{ab}\left(m + \frac{r}{\rho(\sigma)}\right) \delta_{m+n+\frac{r+s}{\rho(\sigma)},0} \right)$$

$$\begin{aligned} &\left[\hat{J}^{(r)}_{aj}\left(m + \frac{r}{\rho(\sigma)}\right), \hat{\bar{J}}^{\sharp}{}_{bl}^{(s)}\left(n + \frac{s}{\rho(\sigma)}\right) \right] = 0 \\ &\hat{\bar{J}}^{\sharp}{}_{aj}^{(r)}\left(m + \frac{r}{\rho(\sigma)}\right) \equiv \hat{\bar{J}}^{(-r)}_{aj}\left(-m - \frac{r}{\rho(\sigma)}\right). \end{aligned} \tag{7.14c}$$

Here the twisted left mover algebra [31, 32] is a semisimple orbifold affine algebra [33], with commuting orbifold affine subalgebras labeled by j, at orbifold affine level $\hat{k} = \rho(\sigma)k$. Moreover, we have used the mode-number reversed relabelling $\hat{\bar{J}}^{\sharp}$ in

(7.14c) to *rectify* the twisted right mover algebra into a copy (7.14b) of the twisted left mover algebra.

We also find that the left and right mover Virasoro generators of $A_g(\mathbb{Z}_\lambda)/\mathbb{Z}_\lambda$ are copies of each other

$$L_\sigma(m) = \frac{1}{2k+Q_{\mathfrak{g}}}\Bigg\{\frac{\eta^{ab}}{\rho(\sigma)}\sum_{r=0}^{\rho(\sigma)-1}\sum_{j=0}^{\frac{\lambda}{\rho(\sigma)}-1}\sum_{p\in\mathbb{Z}} :\hat{J}^{(r)}_{aj}\left(p+\frac{r}{\rho(\sigma)}\right)\hat{J}^{(-r)}_{bj}\left(m-p-\frac{r}{\rho(\sigma)}\right):_M + \\ + \delta_{m,0}\frac{k\lambda\dim\mathfrak{g}}{12}\left(1-\frac{1}{\rho(\sigma)}\right)\Bigg\} \tag{7.15a}$$

$$\bar{L}_\sigma(m) = \frac{1}{2k+Q_{\mathfrak{g}}}\Bigg\{\frac{\eta^{ab}}{\rho(\sigma)}\sum_{r=0}^{\rho(\sigma)-1}\sum_{j=0}^{\frac{\lambda}{\rho(\sigma)}-1}\sum_{p\in\mathbb{Z}} :\hat{\bar{J}}^{\sharp\,(r)}_{aj}\left(p+\frac{r}{\rho(\sigma)}\right)\hat{\bar{J}}^{\sharp\,(-r)}_{bj}\left(m-p-\frac{r}{\rho(\sigma)}\right):_M + \\ + \delta_{m,0}\frac{k\lambda\dim\mathfrak{g}}{12}\left(1-\frac{1}{\rho(\sigma)}\right)\Bigg\} \tag{7.15b}$$

when the right mover Virasoro generators are expressed in terms of the rectified current modes $\hat{\bar{J}}^\sharp$. The left mover result in (7.15a) was given in ref. [31].

The twisted representation matrices of $A_g(\mathbb{Z}_\lambda)/\mathbb{Z}_\lambda$

$$R(\sigma) = \rho(\sigma), \qquad N(r) = n(r) = r, \qquad \bar{r} = 0,\ldots,\rho(\sigma)-1 \tag{7.16a}$$

$$\mathcal{T}^{(r)}_{aj}(T,\sigma) \equiv \mathcal{T}_{raj}(T,\sigma)\,, \qquad \mathcal{T}^{(r)}_{aj}(T,\sigma)_{s\alpha l}{}^{t\beta m} = (T_a)_\alpha{}^\beta\delta_{jl}\delta_l{}^m\delta_{r+s-t,\,0\,mod\,\rho(\sigma)} \tag{7.16b}$$

$$[\mathcal{T}^{(r)}_{aj}(T,\sigma),\mathcal{T}^{(s)}_{bl}(T,\sigma)] = \delta_{jl}if_{ab}{}^c\mathcal{T}^{(r+s)}_{cj}(T,\sigma)\,, \quad \mathcal{T}^{(r\pm\rho(\sigma))}_{aj}(T,\sigma) = \mathcal{T}^{(r)}_{aj}(T,\sigma) \tag{7.16c}$$

also follow from the simplified formula (7.7e) and the data (7.13). The algebra (7.16c) of the twisted representation matrices shows the same semisimplicity as the twisted left and right mover current algebras in (7.14).

Further study of the WZW cyclic permutation orbifolds is found in subsection 9.3.

7.4 The S_N permutation orbifolds

For the permutation group $H = S_N$, it is known [32] that

$$I \longrightarrow \hat{j}j, \quad n(r)l \longrightarrow \hat{l}\,l, \qquad \sum_{j=0}^{n(\vec{\sigma})-1}\sigma_j = N, \qquad \sigma_{j+1} \leqslant \sigma_j \tag{7.17a}$$

$$U^\dagger(\vec{\sigma})_{\hat{j}j}{}^{\hat{l}l} = \frac{\delta_j{}^l}{\sqrt{\sigma_j}}e^{-2\pi i\frac{\hat{j}\hat{l}}{\sigma_j}}, \quad E^j_{\hat{j}}(\vec{\sigma}) = e^{-2\pi i\frac{\hat{j}}{\sigma_j}}, \quad \frac{n(r)}{\rho(\sigma)} = \frac{\hat{j}}{\sigma_j} \tag{7.17b}$$

$$\hat{J}_{n(r)aj} \longrightarrow J^{(\hat{j})}_{aj} \equiv \hat{J}_{\hat{j}aj}, \quad \hat{\bar{J}}_{n(r)aj} \longrightarrow \bar{J}^{(\hat{j})}_{aj} \equiv \hat{\bar{J}}_{\hat{j}aj}, \quad \chi_{\hat{j}aj}(\vec{\sigma}) = \sqrt{\sigma_j} \tag{7.17c}$$

$$\mathcal{G}_{\hat{j}aj;\hat{l}\,bl}(\vec{\sigma}) = \delta_{jl}\sigma_j k\eta_{ab}\delta_{\hat{j}+\hat{l}\,,0\,\mathrm{mod}\,\sigma_j}, \quad \mathcal{G}^{\hat{j}aj;\hat{l}\,bl}(\vec{\sigma}) = \frac{\delta^{jl}}{\sigma_j}\frac{\eta^{ab}}{k}\delta_{\hat{j}+\hat{l}\,,0\,\mathrm{mod}\,\sigma_j} \tag{7.17d}$$

$$\mathcal{F}_{\hat{j}aj;\hat{l}\,bl}{}^{\hat{m}\;cm}(\vec{\sigma}) = f_{ab}{}^{c}\delta_{jl}\delta_{l}{}^{m}\delta_{\hat{j}+\hat{l}\;-\hat{m},0\,\mathrm{mod}\,\sigma_j} \tag{7.17e}$$

$$a = 1\ldots\dim\mathfrak{g}, \qquad j,l = 0,\ldots,n(\vec{\sigma})-1,$$

$$\bar{\hat{j}} = 0,\ldots,\sigma_j-1, \quad \bar{\hat{l}} = 0,\ldots,\sigma_l-1\,. \tag{7.17f}$$

Here each sector $\vec{\sigma} = \{\sigma_j\}$ is a partition of N, which also labels a representative element $h_{\vec{\sigma}}$ of each conjugacy class of S_N: The integer $\hat{j}$ runs within each disjoint cycle σ_j (an element of $\mathbb{Z}_{\sigma_j}$ with size and order σ_j), and the integer j runs over the set of all disjoint cycles of $h_{\vec{\sigma}} \in S_N$. Note that the matrix of eigenvectors $U^{\dagger}(\vec{\sigma})$ in (7.17b) is a direct sum in the space of disjoint cycles.

This leads to the twisted current algebra of the permutation orbifolds $A_g(S_N)/S_N$

$$\left[\hat{J}_{aj}^{(\hat{j})}\left(m+\frac{\hat{j}}{\sigma_j}\right),\hat{J}_{bl}^{(\hat{l})}\left(n+\frac{\hat{l}}{\sigma_l}\right)\right] =$$

$$= \delta_{jl}\left(if_{ab}{}^{c}\hat{J}_{cj}^{(\hat{j}+\hat{l})}\left(m+n+\frac{\hat{j}+\hat{l}}{\sigma_j}\right)+(\sigma_j k)\eta_{ab}\left(m+\frac{\hat{j}}{\sigma_j}\right)\delta_{m+n+\frac{\hat{j}+\hat{l}}{\sigma_j},0}\right) \tag{7.18a}$$

$$\left[\hat{\bar{J}}^{\sharp}_{aj}{}^{(\hat{j})}\left(m+\frac{\hat{j}}{\sigma_j}\right),\hat{\bar{J}}^{\sharp}_{bl}{}^{(\hat{l})}\left(n+\frac{\hat{l}}{\sigma_l}\right)\right] =$$

$$= \delta_{jl}\left(if_{ab}{}^{c}\hat{\bar{J}}^{\sharp}_{cj}{}^{(\hat{j}+\hat{l})}\left(m+n+\frac{\hat{j}+\hat{l}}{\sigma_j}\right)+(\sigma_j k)\eta_{ab}\left(m+\frac{\hat{j}}{\sigma_j}\right)\delta_{m+n+\frac{\hat{j}+\hat{l}}{\sigma_j},0}\right) \tag{7.18b}$$

$$\left[\hat{J}_{aj}^{(\hat{j})}\left(m+\frac{\hat{j}}{\sigma_j}\right),\hat{\bar{J}}^{\sharp}_{bl}{}^{(\hat{l})}\left(n+\frac{\hat{l}}{\sigma_l}\right)\right] = 0 \tag{7.18c}$$

$$\hat{\bar{J}}^{\sharp}_{aj}{}^{(\hat{j})}\left(m+\frac{\hat{j}}{\sigma_j}\right) \equiv \hat{\bar{J}}_{aj}^{(-\hat{j})}\left(-m-\frac{\hat{j}}{\sigma_j}\right). \tag{7.18d}$$

Here the twisted left mover algebra [32] in (7.18a) is a semisimple orbifold affine algebra [33], with commuting subalgebras labeled by j, at orbifold affine level $\hat{k}_j = \sigma_j k$. Moreover, we have used the mode-number reversed relabelling (7.18d) to rectify the twisted right mover algebra into a copy (7.18b) of the twisted left mover algebra. The left and right mover Virasoro generators are also found to be copies (see Subsec. 7.5) when the right mover generators are expressed in terms of the rectified current modes $\hat{\bar{J}}^{\sharp}$.

Finally, we give the twisted representation matrices of $A_g(S_N)/S_N$

$$\frac{N(r)}{R(\sigma)} = \frac{n(r)}{\rho(\sigma)} = \frac{\hat{j}}{\sigma_j}, \qquad \bar{\hat{j}} = 0,\ldots,\sigma_j-1 \tag{7.19a}$$

$$\mathcal{T}_{aj}^{(\hat{j})}(T,\vec{\sigma}) \equiv \mathcal{T}_{\hat{j}aj}(T,\vec{\sigma}), \qquad \mathcal{T}_{aj}^{(\hat{j})}(T,\vec{\sigma})_{\hat{l}\;\alpha l}{}^{\hat{m}\;\beta m} = (T_a)_{\alpha}{}^{\beta}\delta_{jl}\delta_{l}{}^{m}\delta_{\hat{j}+\hat{l}\;-\hat{m}\;,0\,\mathrm{mod}\,\sigma_j} \tag{7.19b}$$

$$[\mathcal{T}_{aj}^{(\hat{j})}(T,\vec{\sigma}),\mathcal{T}_{bl}^{(\hat{l})}(T,\vec{\sigma})] = \delta_{jl}if_{ab}{}^{c}\mathcal{T}_{cj}^{(\hat{j}+\hat{l})}(T,\vec{\sigma}), \qquad \mathcal{T}_{aj}^{(\hat{j}\pm\sigma_j)}(T,\vec{\sigma}) = \mathcal{T}_{aj}^{(\hat{j})}(T,\vec{\sigma}) \tag{7.19c}$$

which also follow from the data (7.17) and eq. (7.7e). The algebra (7.19c) of the twisted representation matrices shows the same semisimplicity as the twisted current algebra (7.18).

Further discussion of the S_N permutation orbifolds can be found in ref. [32].

7.5 Universal forms and rectification of the permutation orbifolds

For all permutation groups $H \subset S_N$ it is well known that a representative element h_σ of each conjugacy class of H can be expressed in terms of disjoint cycles (see e.g. subsection 7.4). The matrix of eigenvectors $U^\dagger(\sigma)$ can always be taken as a direct sum in the space of disjoint cycles, so we find[6] *for all permutation orbifolds* that the twisted left mover current algebra $\hat{\mathfrak{g}}(h_\sigma)$ of sector σ is isomorphic[7] to the following commuting set of orbifold affine algebras:

$$\hat{\mathfrak{g}}(h_\sigma) = \hat{\mathfrak{g}}(H(\text{permutation}) \subset \text{Aut}(g); \sigma) \tag{7.20a}$$

$$\begin{aligned}
&\left[\hat{J}_{n(r)aj}\left(m+\frac{n(r)}{\rho(\sigma)}\right), \hat{J}_{n(r)bl}\left(n+\frac{n(s)}{\rho(\sigma)}\right)\right] = \\
&= \delta_{jl}\Bigg(if_{ab}{}^{c}\hat{J}_{n(r)+n(s),cj}\left(m+n+\frac{n(r)+n(s)}{\rho(\sigma)}\right) + \\
&\qquad + \hat{k}_j(\sigma)\eta_{ab}\left(m+\frac{n(r)}{\rho(\sigma)}\right)\delta_{m+n+\frac{n(r)+n(s)}{\rho(\sigma)},0}\Bigg)
\end{aligned} \tag{7.20b}$$

$$\hat{k}_j(\sigma) \equiv k f_j(\sigma), \qquad a,b,c = 1\ldots\dim\mathfrak{g}, \qquad \sigma = 0,\ldots,N_c-1\,. \tag{7.20c}$$

Here $\{\hat{k}_j(\sigma)\}$ is a set of orbifold affine levels, where $f_j(\sigma)$ is the size and order[8] of each disjoint cycle j in $h_\sigma \in H(\text{permutation})$. The ratios

$$\frac{N(r)}{R(\sigma)} = \frac{n(r)}{\rho(\sigma)} = \frac{\hat{j}}{f_j(\sigma)}, \qquad \hat{\hat{j}} = 0,\ldots,f_j(\sigma)-1 \tag{7.21}$$

are also determined as in subsection 7.4. The twisted right mover current algebra $\hat{\mathfrak{g}}(h_\sigma^{-1})$ commutes with (7.20) and has the same form with $k \to -k$.

Looking back, we see that the mode-number reversed relabellings (7.14c) and (7.18d) rectify each commuting right mover subalgebra separately, and indeed it is easily checked as above that the mode-number reversed relabelling

$$\begin{aligned}
&\left[\hat{\bar{J}}^\sharp_{n(r)aj}\left(m+\frac{n(r)}{\rho(\sigma)}\right), \hat{\bar{J}}^\sharp_{n(r)bl}\left(n+\frac{n(s)}{\rho(\sigma)}\right)\right] = \\
&\qquad = \delta_{jl}\Bigg(if_{ab}{}^{c}\hat{\bar{J}}^\sharp_{n(r)+n(s),cj}\left(m+n+\frac{n(r)+n(s)}{\rho(\sigma)}\right) +
\end{aligned}$$

[6] A qualitative version of this argument was noted in refs. [31] and [32]. See also ref. [39].

[7] That is, the normalization constants $\chi_{n(r)aj}(\sigma)$ can be chosen to obtain the form (7.20b).

[8] We already know that $f_j(\sigma) = \rho(\sigma)$ for $H = \mathbb{Z}_\lambda$ and $f_j(\sigma) = \sigma_j$ for S_N.

$$+\hat{k}_j(\sigma)\eta_{ab}\left(m+\frac{n(r)}{\rho(\sigma)}\right)\,\delta_{m+n+\frac{n(r)+n(s)}{\rho(\sigma)},0}\Bigg) \tag{7.22a}$$

$$\hat{\tilde{J}}^{\sharp}_{n(r)aj}\left(m+\frac{n(r)}{\rho(\sigma)}\right)\equiv\hat{\tilde{J}}_{-n(r),aj}\left(-m-\frac{n(r)}{\rho(\sigma)}\right) \tag{7.22b}$$

provides a *universal rectification* of the twisted right mover algebra ($\hat{\mathfrak{g}}(h_\sigma^{-1})\simeq\mathfrak{g}(h_\sigma)$) for each sector of all permutation orbifolds. In terms of the rectified right mover modes the ground state conditions (7.9b) take the form

$$0=\hat{\tilde{J}}^{\sharp}_{n(r)aj}\left(m+\frac{n(r)}{\rho(\sigma)}\geq 0\right)|0\rangle_\sigma={}_\sigma\langle 0|\hat{\tilde{J}}^{\sharp}_{n(r)aj}\left(m+\frac{n(r)}{\rho(\sigma)}\leq 0\right) \tag{7.23}$$

which are a copy of the twisted left mover conditions in (7.9a).

In the basis (7.20b), we note that the twisted structure constants and twisted metric take the universal form

$$\mathcal{F}_{n(r)aj;n(s)bl}{}^{n(t)cm}(\sigma)=f_{ab}{}^c\delta_{jl}\delta_l^m\delta_{n(r)+n(s)-n(t),0\,\mathrm{mod}\rho(\sigma)} \tag{7.24a}$$

$$\mathcal{G}_{n(r)aj;n(s)bl}(\sigma)=\hat{k}_j(\sigma)\eta_{ab}\delta_{jl}\delta_{n(r)+n(s),0\,\mathrm{mod}\rho(\sigma)} \tag{7.24b}$$

$$\mathcal{F}_{-n(r),aj;-n(s),bl}{}^{-n(t),cm}(\sigma)=\mathcal{F}_{n(r)aj;n(s)bl}{}^{n(t)cm}(\sigma)$$

$$\mathcal{G}_{-n(r),aj;-n(s),bl}(\sigma)=\mathcal{G}_{n(r)aj;n(s)bl}(\sigma) \tag{7.24c}$$

for all sectors of all permutation orbifolds. Moreover, as required by the discussion of subsection 5.4, the universal rectification (7.22) of the permutation orbifolds follows from the symmetries (7.24c) alone.

From (4.12) and (7.24a) we find the universal form of the algebra of the twisted representation matrices

$$[\mathcal{T}_{n(r)aj}(T,\sigma),\mathcal{T}_{n(s)bl}(T,\sigma)]=\delta_{jl}if_{ab}{}^c\mathcal{T}_{n(r)+n(s),cj}(T,\sigma),\qquad a,b,c=1\ldots\dim\mathfrak{g} \tag{7.25}$$

in this basis. The *factorized form* $\mathcal{T}=Tt$ of the twisted representation matrices

$$\mathcal{T}_{n(r)aj}(T,\sigma)=T_a\otimes t_{n(r)j}(\sigma)\equiv T_at_{n(r)j}(\sigma),\qquad t_{n(r)\pm\rho(\sigma),j}(\sigma)=t_{n(r)j}(\sigma) \tag{7.26a}$$

$$[T_a,T_b]=if_{ab}{}^cT_c,\qquad t_{n(r)j}(\sigma)t_{n(s)l}(\sigma)=\delta_{jl}t_{n(r)+n(s),j}(\sigma) \tag{7.26b}$$

$$t_{n(r)j}(\sigma)_{n(s)l}{}^{n(t)m}=\delta_{jl}\delta_l^m\delta_{n(r)+n(s)-n(t),0\ \mathrm{mod}\rho(\sigma)},\qquad\sigma=0,\ldots,N_c-1 \tag{7.26c}$$

is further discussed in subsection 9.3.

Moreover, using (7.7d) and (7.24b), we find the universal forms

$$\mathcal{G}^{n(r)aj;n(s)bl}(\sigma)=\hat{k}_j^{-1}(\sigma)\eta^{ab}\delta^{jl}\delta_{n(r)+n(s),0\,\mathrm{mod}\rho(\sigma)} \tag{7.27a}$$

$$\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)aj;n(s)bl}(\sigma)=\frac{k/\hat{k}_j(\sigma)}{2k+Q_{\mathfrak{g}}}\eta^{ab}\delta^{jl}\delta_{n(r)+n(s),0\,\mathrm{mod}\rho(\sigma)} \tag{7.27b}$$

$$\mathcal{G}^{-n(r),aj;-n(s),bl}(\sigma)=\mathcal{G}^{n(r)aj;n(s)bl}(\sigma)\,,\quad\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{-n(r),aj;-n(s),bl}(\sigma)=\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)aj;n(s)bl}(\sigma) \tag{7.27c}$$

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$$\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)bl;-n(r),aj}(\sigma) = \mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)aj;-n(r),bl}(\sigma)\,. \tag{7.27d}$$

The final symmetry in (7.27d) is sufficient to verify the rectification identity

$$\begin{aligned}
&\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)aj;-n(r),bl}(\sigma) \sum_{p\in\mathbb{Z}} :\hat{\bar{J}}_{n(r)aj}\left(p+\frac{n(r)}{\rho(\sigma)}\right)\hat{\bar{J}}_{-n(r),bl}\left(-m-p+\frac{n(r)}{\rho(\sigma)}\right):_{\bar{M}}= \\
&= \mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)aj;-n(r),bl}(\sigma) \sum_{p\in\mathbb{Z}} :\hat{\bar{J}}^{\sharp}_{n(r)aj}\left(p+\frac{n(r)}{\rho(\sigma)}\right)\hat{\bar{J}}^{\sharp}_{-n(r),bl}\left(m-p+\frac{n(r)}{\rho(\sigma)}\right):_{M}\,,
\end{aligned} \tag{7.28}$$

where $\hat{\bar{J}}^{\sharp}$ are the rectified right mover modes in (7.22b) and M and $\bar{M}$ ordering are defined in (6.4a) and (6.4b).

Moreover, we may use (6.8e), (7.22b) and (7.28) to express the right mover Virasoro generators (7.11b) as a copy

$$\begin{aligned}
\bar{L}_\sigma(m) = \mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)aj;-n(r),bl}(\sigma) \Bigg\{ &\sum_{p\in\mathbb{Z}} :\hat{\bar{J}}^{\sharp}_{n(r)aj}\left(p+\frac{n(r)}{\rho(\sigma)}\right)\hat{\bar{J}}^{\sharp}_{-n(r),bl}\left(m-p-\frac{n(r)}{\rho(\sigma)}\right):_{M} + \\
&+ \delta_{m,0}\frac{\bar{n}(r)}{2\rho(\sigma)}\left(1-\frac{\bar{n}(r)}{\rho(\sigma)}\right)\mathcal{G}_{n(r)aj;-n(r),bl}(\sigma)\Bigg\}
\end{aligned} \tag{7.29}$$

of the left mover generators in (7.11a). It is reasonable to expect more generally (see also subsection 10.2) that this situation will occur whenever the twisted right mover currents are rectifiable.

We finally give the universal form of the classical WZW orbifold action (see subsection 5.7) in the case of WZW permutation orbifolds:

$$\begin{aligned}
&\hat{g}(\mathcal{T},\xi,t,\sigma) = e^{i\hat{\beta}^{n(r)aj}(\xi,t,\sigma)T_a t_{n(r)j}(\sigma)} \\
&\hat{g}(\mathcal{T},\xi,t,\sigma)_{n(r)\alpha j}{}^{n(s)\beta l} = \delta_j^l\ \hat{g}_j(\mathcal{T},\xi,t,\sigma)_{n(r)\alpha}{}^{n(s)\beta} \\
&\hat{g}_j(\mathcal{T},\xi,t,\sigma) = e^{i\sum_{n(r)a}\hat{\beta}^{n(r)aj}(\xi,t,\sigma)T_a\tau_{n(r)}(\sigma)}
\end{aligned} \tag{7.30a}$$

$$\tau_{n(r)}(\sigma)_{n(s)}{}^{n(t)} \equiv \delta_{n(r)+n(s)-n(t),\,0\bmod\rho(\sigma)} \tag{7.30b}$$

$$\hat{\beta}^{n(r)aj}(\xi+2\pi,t,\sigma) = \hat{\beta}^{n(r)aj}(\xi,t,\sigma)e^{2\pi i\frac{n(r)}{\rho(\sigma)}} \tag{7.30c}$$

$$\hat{g}(\mathcal{T},\xi+2\pi,t,\sigma)_{n(r)\alpha j}{}^{n(s)\beta l} = e^{-2\pi i\frac{n(r)-n(s)}{\rho(\sigma)}}\hat{g}(\mathcal{T},\xi,t,\sigma)_{n(r)\alpha j}{}^{n(s)\beta l} \tag{7.30d}$$

$$\hat{g}_j(\mathcal{T},\xi+2\pi,t,\sigma)_{n(r)\alpha}{}^{n(s)\beta} = e^{-2\pi i\frac{n(r)-n(s)}{\rho(\sigma)}}\hat{g}_j(\mathcal{T},\xi,t,\sigma)_{n(r)\alpha}{}^{n(s)\beta} \tag{7.30e}$$

$$\begin{aligned}
\hat{S}_\sigma[\mathcal{M},\hat{g}] = -\frac{k}{y(T)}\sum_j\Bigg(&\frac{1}{8\pi}\int d^2\xi\ \widehat{\mathrm{Tr}}\big(\hat{g}_j^{-1}(\mathcal{T},\sigma)\partial_+\hat{g}_j(\mathcal{T},\sigma)\ \hat{g}_j^{-1}(\mathcal{T},\sigma)\partial_-\hat{g}_j(\mathcal{T},\sigma)\ \big) + \\
&+\frac{1}{12\pi}\int_\Gamma \widehat{\mathrm{Tr}}\big(\,(\ \hat{g}_j^{-1}(\mathcal{T},\sigma)d\hat{g}_j(\mathcal{T},\sigma)\ )^3\,\big)\Bigg)
\end{aligned} \tag{7.30f}$$

$$\widehat{\mathrm{Tr}}(AB) \equiv \sum_{n(r),n(s)}\sum_{\alpha,\beta} A_{n(r)\alpha}{}^{n(s)\beta}B_{n(s)\beta}{}^{n(r)\alpha}\,. \tag{7.30g}$$

In obtaining these results, we used eqs. (5.53), (5.55a), (7.7f) and (7.26). Note that, as it should be, the orbifold action is separable in the semisimplicity index j of the universal orbifold affine algebra (7.20). The required integers and the range of j are given explicitly for $A_g(\mathbb{Z}_\lambda)/\mathbb{Z}_\lambda$ and $A_g(S_N)/S_N$ in subsections 7.3 and 7.4.

8. The twisted KZ equations of WZW permutation orbifolds

8.1 Preliminaries

For the WZW permutation orbifolds the vertex operator equations (6.13) take the form

$$\begin{aligned}\partial\hat{g}_+(\mathcal{T},z) &= 2\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)aj;-n(r),bl}(\sigma)\times\\ &\quad\times\left(:\hat{J}_{n(r)aj}(z)\hat{g}_+(\mathcal{T},z):_M-\frac{\bar{n}(r)}{\rho(\sigma)}\frac{1}{z}\hat{g}_+(\mathcal{T},z)\mathcal{T}_{n(r)aj}\right)\mathcal{T}_{-n(r),bl} \qquad (8.1a)\\ \bar{\partial}\hat{g}_-(\mathcal{T},\bar{z}) &= -2\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)aj;-n(r),bl}(\sigma)\mathcal{T}_{-n(r),bl}\times\\ &\quad\times\left(:\hat{\bar{J}}_{n(r)aj}(z)\hat{g}_-(\mathcal{T},\bar{z}):_{\bar{M}}+\frac{\overline{-n(r)}}{\rho(\sigma)}\frac{1}{\bar{z}}\mathcal{T}_{n(r)aj}\hat{g}_-(\mathcal{T},\bar{z})\right). \qquad (8.1b)\end{aligned}$$

Similarly, we obtain the alternate form of the vertex operator equations

$$\begin{aligned}\partial\hat{g}_+(\mathcal{T},z) &= \mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)aj;-n(r),bl}(\sigma)\Bigg(2:\hat{J}_{n(r)aj}(z)\hat{g}_+(\mathcal{T},z):_M\mathcal{T}_{-n(r),bl}+\\ &\qquad+\frac{1}{z}\hat{g}_+(\mathcal{T},z)\mathcal{T}_{n(r)aj}\mathcal{T}_{-n(r),bl}(1-\delta_{\bar{n}(r),0})\Bigg) \qquad (8.2a)\\ \bar{\partial}\hat{g}_-(\mathcal{T},\bar{z}) &= -\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)aj;-n(r),bl}(\sigma)\Bigg(2\mathcal{T}_{-n(r),bl}:\hat{\bar{J}}_{n(r)aj}(z)\hat{g}_-(\mathcal{T},\bar{z}):_{\bar{M}}\\ &\qquad+\frac{1}{\bar{z}}\mathcal{T}_{-n(r),bl}\mathcal{T}_{n(r)aj}\hat{g}_-(\mathcal{T},\bar{z})(1-\delta_{\bar{n}(r),0})\Bigg), \qquad (8.2b)\end{aligned}$$

where we have used (7.7c) to show that the $\mathcal{F}$-terms of (6.19) are zero for WZW permutation orbifolds. Correspondingly, the identities (6.20) take the form

$$\begin{aligned}2\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)aj;-n(r),bl}(\sigma)\frac{\bar{n}(r)}{\rho(\sigma)}\mathcal{T}_{n(r)aj}\mathcal{T}_{-n(r),bl} &= \mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)aj;-n(r),bl}(\sigma)\mathcal{T}_{n(r)aj}\mathcal{T}_{-n(r),bl}(1-\delta_{\bar{n}(r),0})\\ 2\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)aj;-n(r),bl}(\sigma)\frac{\overline{-n(r)}}{\rho(\sigma)}\mathcal{T}_{-n(r),bl}\mathcal{T}_{n(r)aj} &= \mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)aj;-n(r),bl}(\sigma)\mathcal{T}_{-n(r),bl}\mathcal{T}_{n(r)aj}(1-\delta_{\bar{n}(r),0}) \qquad (8.3a)\\ &= \mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)aj;-n(r),bl}(\sigma)\mathcal{T}_{n(r)bl}\mathcal{T}_{-n(r),aj}(1-\delta_{\bar{n}(r),0}) \qquad (8.3b)\end{aligned}$$

for the WZW permutation orbifolds. The last form in (8.3b) follows by $n\to -n$ from (8.3a).

Using the ground state conditions (7.9), we find that the twisted partial currents obey

$$0 = \hat{J}^{+}_{n(r)aj}(z)|0\rangle_\sigma = {}_\sigma\langle 0|\hat{J}^{-}_{n(r)aj}(z) = {}_\sigma\langle 0| :\hat{g}_+(\mathcal{T},z)\hat{J}_{n(r)aj}(z):_M |0\rangle_\sigma \tag{8.4a}$$

$$0 = \hat{\bar{J}}^{-}_{n(r)aj}(\bar{z})|0\rangle_\sigma = {}_\sigma\langle 0|\hat{\bar{J}}^{+}_{n(r)aj}(\bar{z}) = {}_\sigma\langle 0| :\hat{g}_-(\mathcal{T},\bar{z})\hat{\bar{J}}_{n(r)aj}(\bar{z}):_{\bar{M}} |0\rangle_\sigma\,, \tag{8.4b}$$

where we used (6.14) and (6.15) for the left and right movers respectively.

8.2 The twisted left mover KZ equations

Using the twisted vertex operator equation (8.1a), the ground state conditions (8.4a) and the commutation relations (6.17), we obtain after some algebra the *twisted Knizhnik-Zamolodchikov equations*

$$\frac{A_g(H)}{H}, \qquad H(\text{permutation}) \subset \mathrm{Aut}(g) \tag{8.5a}$$

$$\hat{A}_+(\mathcal{T},z,\sigma) \equiv \langle \hat{g}_+(\mathcal{T}^{(1)},z_1,\sigma)\hat{g}_+(\mathcal{T}^{(2)},z_2,\sigma)\cdots\hat{g}_+(\mathcal{T}^{(N)},z_N,\sigma)\rangle_\sigma \tag{8.5b}$$

$$\partial_\mu \hat{A}_+(\mathcal{T},z,\sigma) = \hat{A}_+(\mathcal{T},z,\sigma)\hat{W}_\mu(\mathcal{T},z,\sigma), \qquad \sigma = 0,\ldots,N_c-1 \tag{8.5c}$$

$$\hat{W}_\mu(\mathcal{T},z,\sigma) = 2\mathcal{L}^{n(r)aj;-n(r),bl}_{\hat{\mathfrak{g}}(\sigma)}(\sigma)\times \\ \times\left[\sum_{\nu\neq\mu}\left(\frac{z_\nu}{z_\mu}\right)^{\frac{\bar{n}(r)}{\rho(\sigma)}}\frac{1}{z_{\mu\nu}}\mathcal{T}^{(\nu)}_{n(r)aj}\mathcal{T}^{(\mu)}_{-n(r),bl} - \frac{\bar{n}(r)}{\rho(\sigma)}\frac{1}{z_\mu}\mathcal{T}^{(\mu)}_{n(r)aj}\mathcal{T}^{(\mu)}_{-n(r),bl}\right] \tag{8.5d}$$

$$\mathcal{T}^{(\nu)}\mathcal{T}^{(\mu)} \equiv \mathcal{T}^{(\nu)}\otimes\mathcal{T}^{(\mu)}, \qquad z_{\mu\nu}\equiv z_\mu - z_\nu, \qquad \sum_{\nu\neq\mu} \equiv \sum_{\nu\neq\mu\ \nu=1}^{N} \tag{8.5e}$$

for each twisted left mover sector σ of all WZW permutation orbifolds. These equations are a central result of this paper. We remind the reader of the data in subsections 7.3 and 7.4 and the universal forms of $\mathcal{T} = Tt$ and $\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}(\sigma)$ in eqs. (7.26) and (7.27) respectively (see also subsections 9.3).

By construction the twisted connection (8.5d) is flat

$$\partial_\mu \hat{W}_\nu(\sigma) - \partial_\nu \hat{W}_\mu(\sigma) + [\hat{W}_\mu(\sigma),\hat{W}_\nu(\sigma)] = 0\,, \qquad \hat{W}_\mu(\sigma)\equiv \hat{W}_\mu(\mathcal{T},z,\sigma) \tag{8.6}$$

and in fact this twisted connection is abelian flat

$$\partial_\mu \hat{W}_\nu(\sigma) - \partial_\nu \hat{W}_\mu(\sigma) = [\hat{W}_\mu(\sigma),\hat{W}_\nu(\sigma)] = 0\,. \tag{8.7}$$

To check abelian flatness, compute the derivatives explicitly. For $\bar{n}(r) = 0$ the identity is immediate. For $\bar{n}(r)\neq 0$ use in the second term the symmetry (4.24b) of $\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}$ and relabel $n\to -n$ so that $\frac{\bar{n}}{\rho}\to 1-\frac{\bar{n}}{\rho}$. Then the two terms sum to zero. The condition (8.7) is checked explicitly for the cyclic permutation orbifolds in subsection 9.3. The z_μ^{-1} terms in (8.5) can be traced back to the fact that the twist field states $|0\rangle_\sigma = \tau_\sigma(0)|0\rangle$ are not invariant under $Sl(2)\oplus Sl(2)$.

For completeness, we also give the alternate form of the twisted connection

$$\hat{W}_\mu(\sigma) = \mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)aj;-n(r),bl}(\sigma) \times \qquad (8.8)$$

$$\times \left[\sum_{\nu\neq\mu} \left(\frac{z_\nu}{z_\mu}\right)^{\frac{\bar{n}(r)}{\rho(\sigma)}} \frac{2}{z_{\mu\nu}} \mathcal{T}^{(\nu)}_{n(r)aj} \mathcal{T}^{(\mu)}_{-n(r),bl} - (1-\delta_{\bar{n}(r),0}) \frac{1}{z_\mu} \mathcal{T}^{(\mu)}_{n(r)aj} \mathcal{T}^{(\mu)}_{-n(r),bl} \right]$$

that follows from the alternate form (8.2a) of the twisted vertex operator equation.

We also find global Ward identities corresponding to the *residual symmetry* generated by the elements of the Lie subalgebra (5.18) of sector σ. For the left mover sectors of the WZW permutation orbifolds, these Ward identities read

$$\langle [\hat{J}_{0aj}(0), \hat{g}_+(\mathcal{T}^{(1)}, z_1, \sigma) \cdots\cdots \hat{g}_+(\mathcal{T}^{(N)}, z_N, \sigma)] \rangle_\sigma = 0 \;\Rightarrow\; \hat{A}_+(\mathcal{T}, z, \sigma) \hat{Q}_{aj}(\sigma) = 0$$

$$\hat{Q}_{aj}(\sigma) \equiv \sum_{\mu=1}^{N} \mathcal{T}^{(\mu)}_{0aj}\,,\; [\hat{Q}_{aj}(\sigma), \hat{Q}_{bl}(\sigma)] = i\mathcal{F}_{0aj;0bl}{}^{0cm} \hat{Q}_{cm}(\sigma)\,,\; a = 1\ldots \dim \mathfrak{g} \;\forall j \qquad (8.9)$$

in terms of the twisted representation matrices $\mathcal{T}$ with $n(r) = 0$. Using the properties (7.26) of the twisted representation matrices, we also find that $\hat{W}_\mu$ commutes with $\hat{Q}_{aj}$ so that the global Ward identities (8.9) are consistent with the twisted KZ equations.

Finally, we note the simple form of the $L_\sigma(0)$ Ward identities

$$\hat{A}_+(\mathcal{T}, z, \sigma) \sum_{\mu=1}^{N} \left(\overleftarrow{\partial}_\mu z_\mu + \Delta_{\mathfrak{g}}(T^{(\mu)}) \right) = 0 \qquad (8.10)$$

which follows from (5.39) and (7.7g) for the WZW permutation orbifolds. In fact (as in the untwisted case) we find that the $L_\sigma(0)$ Ward identities are automatically satisfied when both the twisted KZ equations (8.5) and the global Ward identities (8.9) are satisfied. To see this, the reader may find helpful the following intermediate steps:

$$\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)aj;-n(r),bl}(\sigma) \sum_{\mu,\nu\;\mu\neq\nu} \frac{z_\mu}{z_{\mu\nu}} \left(\frac{z_\nu}{z_\mu}\right)^{\frac{\bar{n}(r)}{\rho(\sigma)}} \mathcal{T}^{(\nu)}_{n(r)aj} \mathcal{T}^{(\mu)}_{-n(r),bl} (1-\delta_{\bar{n}(r),0}) = 0 \qquad (8.11a)$$

$$\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{0aj;0bl}(\sigma) \sum_{\mu,\nu\;\mu\neq\nu} \frac{z_\nu}{z_{\mu\nu}} \mathcal{T}^{(\nu)}_{0aj} \mathcal{T}^{(\mu)}_{0bl} = -\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{0aj;0bl}(\sigma) \sum_{\mu,\nu\;\mu\neq\nu} \frac{z_\mu}{z_{\mu\nu}} \mathcal{T}^{(\nu)}_{0aj} \mathcal{T}^{(\mu)}_{0bl} \qquad (8.11b)$$

$$\sum_\mu z_\mu \hat{W}_\mu(\sigma) = \mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{0aj;0bl}(\sigma) \sum_{\mu,\nu\;\mu\neq\nu} \mathcal{T}^{(\nu)}_{0aj} \mathcal{T}^{(\mu)}_{0bl} - $$

$$- \mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)aj;-n(r),bl}(\sigma) \sum_\mu \mathcal{T}^{(\mu)}_{n(r)aj} \mathcal{T}^{(\mu)}_{-n(r),bl} (1-\delta_{\bar{n}(r),0}) \qquad (8.11c)$$

$$\hat{A}_+(\sigma)\; \mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{0aj;0bl}(\sigma) \sum_{\mu,\nu\;\mu\neq\nu} \mathcal{T}^{(\nu)}_{0aj} \mathcal{T}^{(\mu)}_{0bl} = -\hat{A}_+(\sigma)\; \mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{0aj;0bl}(\sigma) \sum_\mu \mathcal{T}^{(\mu)}_{0aj} \mathcal{T}^{(\mu)}_{0bl} \qquad (8.11d)$$

$$\hat{A}_+(\sigma)\sum_\mu z_\mu \hat{W}_\mu(\sigma) = -\hat{A}_+(\sigma)\ \mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)aj;-n(r),bl}(\sigma)\sum_\mu \mathcal{T}^{(\mu)}_{n(r)aj}\mathcal{T}^{(\mu)}_{-n(r),bl}$$
$$= -\hat{A}_+(\sigma)\sum_\mu \Delta_{\mathfrak{g}}(T^{(\mu)})\ . \tag{8.11e}$$

The first identity in (8.11) holds for each $(\mu\nu)+(\nu\mu)$ pair of terms separately, without further summation. To see this use $n \to -n$ in the second term. The second identity follows on $\mu \leftrightarrow \nu$ exchange, as in the untwisted case. The third identity then follows immediately from the alternate form (8.8) of the twisted KZ connection. The final identities follow from the global Ward identities and (7.7g).

We finally recall from (7.2) that for $\sigma = 0$

$$\bar{n}(r) = 0, \qquad j,l \to J,L = 0\ldots,K-1, \qquad \mathcal{L}_{\mathfrak{g}(0)}(0) \to L_g^{aJ,bL}, \qquad \mathcal{T}(0) \to T_{aJ} \tag{8.12}$$

so our twisted KZ system reduces to the appropriate untwisted KZ system in the untwisted sector of each permutation orbifold.

8.3 The twisted right mover KZ equations

For the twisted right mover sectors, the derivation proceeds along the same lines. Using the twisted vertex operator equation (8.1b), the ground state conditions (8.4b) and the commutation relations (6.17), we obtain the twisted KZ equations for the right movers

$$\hat{A}_-(\sigma) \equiv \hat{A}_-(\mathcal{T},\bar{z},\sigma) = \langle \hat{g}_-(\mathcal{T}^{(1)},\bar{z}_1,\sigma)\hat{g}_-(\mathcal{T}^{(2)},\bar{z}_2,\sigma)\cdots\hat{g}_-(\mathcal{T}^{(N)},\bar{z}_N,\sigma)\rangle_\sigma \tag{8.13a}$$
$$\bar{\partial}_\mu \hat{A}_-(\sigma) = \hat{\bar{W}}_\mu(\sigma)\hat{A}_-(\sigma), \qquad \hat{\bar{W}}_\mu(\sigma) \equiv \hat{\bar{W}}_\mu(\mathcal{T},\bar{z},\sigma), \qquad \sigma = 0,\cdots,N_c-1 \tag{8.13b}$$
$$\hat{\bar{W}}_\mu(\sigma) = 2\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)aj;-n(r),bl}(\sigma)\ \times \tag{8.13c}$$
$$\times\left[\sum_{\nu\neq\mu}\left(\frac{\bar{z}_\nu}{\bar{z}_\mu}\right)^{\overline{\frac{-n(r)}{\rho(\sigma)}}}\frac{1}{\bar{z}_{\mu\nu}}\mathcal{T}^{(\mu)}_{-n(r),bl}\mathcal{T}^{(\nu)}_{n(r)aj} - \frac{\overline{-n(r)}}{\rho(\sigma)}\frac{1}{\bar{z}_\mu}\mathcal{T}^{(\mu)}_{-n(r),bl}\mathcal{T}^{(\mu)}_{n(r)aj}\right],$$

where $\bar{z}_{\mu\nu} \equiv \bar{z}_\mu - \bar{z}_\nu$. By construction the twisted right mover connection $\hat{\bar{W}}(\sigma)$ is also flat

$$\bar{\partial}_\mu \hat{\bar{W}}_\nu(\sigma) - \bar{\partial}_\nu \hat{\bar{W}}_\mu(\sigma) - [\hat{\bar{W}}_\mu(\sigma),\hat{\bar{W}}_\nu(\sigma)] = 0 \tag{8.14}$$

and it is easily checked as above that this connection is also abelian flat. Note also the alternate form of the right mover connection

$$\hat{\bar{W}}_\mu(\sigma) = \mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)aj;-n(r),bl}(\sigma)\ \times \tag{8.15}$$
$$\times\left[\sum_{\nu\neq\mu}\left(\frac{\bar{z}_\nu}{\bar{z}_\mu}\right)^{\frac{\bar{n}(r)}{\rho(\sigma)}}\frac{2}{\bar{z}_{\mu\nu}}\mathcal{T}^{(\mu)}_{n(r)aj}\mathcal{T}^{(\nu)}_{-n(r),bl} - (1-\delta_{\bar{n}(r),0})\frac{1}{\bar{z}_\mu}\mathcal{T}^{(\mu)}_{n(r)aj}\mathcal{T}^{(\mu)}_{-n(r),bl}\right].$$

This form follows using (8.2b), a relabelling $n \to -n$ and the identity (8.3b). The alternate form shows that the twisted right mover connection differs from the twisted left mover connection (8.8) only by $z_\mu \to \bar{z}_\mu$ and reversal of the order of the twisted representation matrices $\mathcal{T}$.

Finally, the Ward identities for the twisted right mover correlators

$$\langle[\hat{\bar{J}}_{0aj}(0), \hat{g}_-(\mathcal{T}^{(1)}, \bar{z}_1, \sigma)\cdots\hat{g}_-(\mathcal{T}^{(N)}, \bar{z}_N, \sigma)]\rangle_\sigma = 0 \quad \Rightarrow \quad \hat{Q}_{aj}(\sigma)\hat{A}_-(\sigma) = 0 \tag{8.16a}$$

$$\sum_{\mu=1}^{N}\left(\bar{z}_\mu\bar{\partial}_\mu + \Delta_{\mathfrak{g}}(T^{(\mu)})\right)\hat{A}_-(\sigma) = 0 \tag{8.16b}$$

are obtained from (5.18), (5.20b) and (5.39). Moreover, one finds again that the $\bar{L}_\sigma(0)$ Ward identity (8.16b) is satisfied given the twisted right mover KZ equation (8.13) and the global Ward identities in (8.16a).

9. Examples in the WZW permutation orbifolds

9.1 General one-point correlators

The twisted KZ equation for the left mover one-point correlators

$$\partial\langle\hat{g}_+(\mathcal{T}, z, \sigma)\rangle_\sigma = -\frac{2}{z}\langle\hat{g}_+(\mathcal{T}, z, \sigma)\rangle_\sigma \mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)aj;-n(r),bl}(\sigma)\tfrac{\bar{n}(r)}{\rho(\sigma)}\mathcal{T}_{n(r)aj}\mathcal{T}_{-n(r),bl} \tag{9.1a}$$

$$\langle\hat{g}_+(\mathcal{T}, z, \sigma)\rangle_\sigma\mathcal{T}_{0aj} = 0, \qquad a = 1\ldots\dim\mathfrak{g}, \qquad \forall\, j \tag{9.1b}$$

follows from eq. (8.5), and we have included the global Ward identity in (9.1b). The solution of this system is

$$\begin{aligned}\langle\hat{g}_+(\mathcal{T}, z, \sigma)\rangle_\sigma &= C_+(\mathcal{T}, \sigma)z^{-2\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)aj;-n(r),bl}(\sigma)\frac{\bar{n}(r)}{\rho(\sigma)}\mathcal{T}_{n(r)aj}\mathcal{T}_{-n(r),bl}} && (9.2a)\\ &= C_+(\mathcal{T}, \sigma)z^{-\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{n(r)aj;-n(r),bl}(\sigma)\mathcal{T}_{n(r)aj}\mathcal{T}_{-n(r),bl}(1-\delta_{\bar{n}(r),0})} && (9.2b)\\ &= C_+(\mathcal{T}, \sigma)z^{-\Delta_{\mathfrak{g}}(T)+\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{0aj;0bl}(\sigma)\mathcal{T}_{0aj}\mathcal{T}_{0bl}} && (9.2c)\\ C_+(\mathcal{T}, \sigma)\mathcal{T}_{0aj} &= 0, \qquad a = 1\ldots\dim\mathfrak{g}\,, \forall\, j\,, && (9.2d)\end{aligned}$$

where we have used (6.20a) to obtain (9.2b) and (7.7g) to obtain (9.2c).

On the other hand, the $L_\sigma(0)$ Ward identities (8.10) read in this case

$$\langle\hat{g}_+(\mathcal{T}, z, \sigma)\rangle_\sigma(\overleftarrow{\partial_z}z + \Delta_{\mathfrak{g}}(T)) = 0 \quad \Rightarrow \quad \langle\hat{g}_+(\mathcal{T}, z, \sigma)\rangle_\sigma \propto z^{-\Delta_{\mathfrak{g}}(T)}\,. \tag{9.3}$$

The solutions in (9.2) and (9.3) agree iff

$$C_+(\mathcal{T}, \sigma)\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{0aj;0bl}(\sigma)\mathcal{T}_{0aj}\mathcal{T}_{0bl} = 0 \tag{9.4}$$

which also follows directly from eq. (9.2d). But in fact the quadratic structure in (9.4) is not in general zero (see e.g. subsection 9.3)

$$\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{0aj;0bl}(\sigma)\mathcal{T}_{0aj}\mathcal{T}_{0bl} \neq 0 \tag{9.5}$$

which implies that the twisted one-point correlators vanish

$$C_+(\mathcal{T},\sigma)=0 \quad \Rightarrow \quad \langle \hat{g}_+(\mathcal{T},z,\sigma)\rangle_\sigma=0, \qquad \sigma=0,\ldots,N_c-1 \tag{9.6}$$

for non-trivial $\mathcal{T}$ for all sectors of all WZW permutation orbifolds.

Another way to understand the vanishing of the one-point correlators is to note that eq. (9.2d) has no non-trivial solution. To see this, one may peel off unitary matrices from $\mathcal{T}$ in (7.7e) to show that (9.2d) is equivalent to the simpler problem for the untwisted representation matrices T

$$c_+(T)^\beta (T_a)_\beta{}^\alpha=0, \qquad a=1\ldots\dim g \tag{9.7}$$

which also has no non-trivial solution. This situation is familiar in the case of the *untwisted* one-point WZW correlators, where the global Ward identity $c_+T_a=0,\ \forall\, a$ has no non-trivial solution.

Similarly, one finds $\langle \hat{g}_-(\mathcal{T},\bar{z},\sigma)\rangle_\sigma=0$ for the twisted one-point right mover correlators with non-trivial $\mathcal{T}$.

9.2 General left mover two-point correlators

In this subsection, we solve the twisted KZ equations (8.5) for the left mover two-point correlators

$$\hat{A}_+(1,2)\equiv\langle \hat{g}(\mathcal{T}^{(1)},z_1,\sigma)\hat{g}(\mathcal{T}^{(2)},z_2,\sigma)\rangle_\sigma \tag{9.8}$$

of the general WZW permutation orbifold. In this case the global Ward identity reads

$$\hat{A}_+(1,2)(\mathcal{T}^{(1)}_{0aj}+\mathcal{T}^{(2)}_{0aj})=0, \qquad a=1\ldots\dim\mathfrak{g}\,,\forall\, j \tag{9.9}$$

and this tells us that

$$\mathcal{L}^{0aj;0bl}_{\hat{\mathfrak{g}}(\sigma)}(\sigma)\mathcal{T}^{(1)}_{0aj}\mathcal{T}^{(1)}_{0bl}=\mathcal{L}^{0aj;0bl}_{\hat{\mathfrak{g}}(\sigma)}(\sigma)\mathcal{T}^{(2)}_{0aj}\mathcal{T}^{(2)}_{0bl} \qquad \text{when }\ \hat{A}_+(1,2)\neq 0 \tag{9.10}$$

because of the symmetry (4.24b) of $\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}(\sigma)$.

For purposes of integration, the optimal form for the connections of the two-point correlators is

$$\begin{aligned}\hat{W}_1 \simeq & -\frac{2}{z_{12}}\mathcal{L}^{0aj;0bl}_{\hat{\mathfrak{g}}(\sigma)}(\sigma)\mathcal{T}^{(1)}_{0aj}\mathcal{T}^{(1)}_{0bl}+\mathcal{L}^{n(r)aj;-n(r),bl}_{\hat{\mathfrak{g}}(\sigma)}(\sigma)\times \\ & \times\left[\frac{2}{z_{12}}\left(\frac{z_2}{z_1}\right)^{\frac{\bar{n}(r)}{\rho(\sigma)}}\mathcal{T}^{(2)}_{n(r)aj}\mathcal{T}^{(1)}_{-n(r),bl}-\frac{1}{z_1}\mathcal{T}^{(1)}_{n(r)aj}\mathcal{T}^{(1)}_{-n(r),bl}\right](1-\delta_{\bar{n}(r),0}) \qquad (9.11\text{a}) \\ \hat{W}_2 \simeq & -\frac{2}{z_{21}}\mathcal{L}^{0aj;0bl}_{\hat{\mathfrak{g}}(\sigma)}(\sigma)\mathcal{T}^{(1)}_{0aj}\mathcal{T}^{(1)}_{0bl}+\mathcal{L}^{n(r)aj;-n(r),bl}_{\hat{\mathfrak{g}}(\sigma)}(\sigma)\times\end{aligned}$$

$$\times \left[\frac{2}{z_{21}} \left(\frac{z_1}{z_2} \right)^{1-\frac{\bar{n}(r)}{\rho(\sigma)}} \mathcal{T}^{(2)}_{n(r)aj} \mathcal{T}^{(1)}_{-n(r),bl} - \frac{1}{z_2} \mathcal{T}^{(2)}_{n(r)aj} \mathcal{T}^{(2)}_{-n(r),bl} \right] (1 - \delta_{\bar{n}(r),0}) \, , \tag{9.11b}$$

where we have used (8.8), (4.24b) and the global Ward identity (8.9), including the form (9.10). We have also used $n(r) \to -n(r)$ and (4.24b) in the second term of $\hat{W}_2$. Using eqs. (7.7b), (7.7d) and (7.26), all the matrix structures in (9.11) are seen to commute.

This gives the solution for the general two-point correlators

$$\begin{aligned}
\hat{A}_+(1,2) = {} & C_+(\mathcal{T},\sigma) z_1^{-\mathcal{L}^{n(r)aj;-n(r),bl}_{\hat{\mathfrak{g}}(\sigma)}(\sigma) \mathcal{T}^{(1)}_{n(r)aj} \mathcal{T}^{(1)}_{-n(r),bl} (1-\delta_{\bar{n}(r),0})} \times \\
& \times z_2^{-\mathcal{L}^{n(r)aj;-n(r),bl}_{\hat{\mathfrak{g}}(\sigma)}(\sigma) \mathcal{T}^{(2)}_{n(r)aj} \mathcal{T}^{(2)}_{-n(r),bl} (1-\delta_{\bar{n}(r),0})} \times \\
& \times z_{12}^{-2\mathcal{L}^{0aj;0bl}_{\hat{\mathfrak{g}}(\sigma)}(\sigma) \mathcal{T}^{(1)}_{0aj} \mathcal{T}^{(1)}_{0bl}} \times \\
& \times \exp\left(2\mathcal{L}^{n(r)aj;-n(r),bl}_{\hat{\mathfrak{g}}(\sigma)}(\sigma) I_{\frac{\bar{n}(r)}{\rho(\sigma)}}(\tfrac{z_1}{z_2}) \mathcal{T}^{(2)}_{n(r)aj} \mathcal{T}^{(1)}_{-n(r),bl} (1-\delta_{\bar{n}(r),0}) \right)
\end{aligned} \tag{9.12a}$$

$$\begin{aligned}
= {} & C_+(\mathcal{T},\sigma) z_1^{-\Delta_{\mathfrak{g}}(T^{(1)})} z_2^{-\Delta_{\mathfrak{g}}(T^{(2)})} \left(\frac{z_1 z_2}{z_{12}^2} \right)^{\mathcal{L}^{0aj;0bl}_{\hat{\mathfrak{g}}(\sigma)}(\sigma) \mathcal{T}^{(1)}_{0aj} \mathcal{T}^{(1)}_{0bl}} \times \\
& \times \exp\left(2\mathcal{L}^{n(r)aj;-n(r),bl}_{\hat{\mathfrak{g}}(\sigma)}(\sigma) I_{\frac{\bar{n}(r)}{\rho(\sigma)}}(\tfrac{z_1}{z_2}) \mathcal{T}^{(2)}_{n(r)aj} \mathcal{T}^{(1)}_{-n(r),bl} (1-\delta_{\bar{n}(r),0}) \right)
\end{aligned} \tag{9.12b}$$

$$C_+(\mathcal{T},\sigma)(\mathcal{T}^{(1)}_{0aj} + \mathcal{T}^{(2)}_{0aj}) = 0, \qquad \sigma = 0, \ldots, N_c - 1 \, . \tag{9.12c}$$

Here we have used (9.10) and (7.7g) to obtain the form in (9.12b), and the integrals

$$I_{\frac{\bar{n}(r)}{\rho(\sigma)}}(y) \equiv \int^y \frac{dx}{x-1} x^{-\frac{\bar{n}(r)}{\rho(\sigma)}} \tag{9.13}$$

are evaluated in appendix B.

As a check, we note that the result (9.12) satisfies the $L_\sigma(0)$ Ward identity

$$\hat{A}_+(1,2) \big(\overleftarrow{\partial}_{z_1} z_1 + \overleftarrow{\partial}_{z_2} z_2 + \Delta_{\mathfrak{g}}(T^{(1)}) + \Delta_{\mathfrak{g}}(T^{(2)}) \big) = 0 \tag{9.14}$$

because $I_{\frac{\bar{n}(r)}{\rho(\sigma)}}(\frac{z_1}{z_2})$ is scale invariant.

9.3 More about the cyclic permutation orbifolds

The twisted KZ systems of subsections 8.2 and 8.3 can be worked out more explicitly by using the universal forms of $\mathcal{T} = Tt$ and $\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}$ in eqs. (7.26) and (7.27) respectively. In this subsection we shall do so for the case of the general WZW cyclic permutation orbifold $A_g(\mathbb{Z}_\lambda)/\mathbb{Z}_\lambda$, whose data is also given in subsection 7.3. We need in particular the twisted representation matrices (7.16b) in the factorized form

$$\mathcal{T}^{(r)}_{aj}(T,\sigma) \equiv \mathcal{T}^{(r)}_{aj} = T_a \otimes t_{rj} \equiv T_a t_{rj}, \qquad [T_a, T_b] = i f_{ab}{}^c T_c \tag{9.15a}$$

$$(t_{rj})_{sl}{}^{tm} = \delta_{jl}\delta_l^m \delta_{r+s-t,0 \bmod \rho(\sigma)}, \qquad t_{rj}t_{sl} = \delta_{jl}t_{r+s,j}, \qquad t_{r\pm\rho(\sigma),j} = t_{rj} \tag{9.15b}$$

$$a = 1 \ldots \dim \mathfrak{g}, \quad \bar{r} = 0, \ldots, \rho(\sigma)-1,$$

$$j = 0, \ldots, \tfrac{\lambda}{\rho(\sigma)} - 1, \quad \sigma = 0, \ldots, \lambda - 1 \tag{9.15c}$$

which is a special case of eq. (7.26).

Using (7.13d), (6.20a) and (9.15), we may evaluate the quadratic forms

$$\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{raj;-r,bl}(\sigma)\mathcal{T}_{aj}^{(r)}\mathcal{T}_{bl}^{(-r)} = \Delta_{\mathfrak{g}}(T)1\!\!1, \qquad (1\!\!1)_{s\alpha l}{}^{t\beta m} \equiv \delta_{s-t,0 \bmod \rho(\sigma)}\delta_\alpha^\beta \delta_l^m \tag{9.16a}$$

$$\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{0aj;0bl}(\sigma)\mathcal{T}_{aj}^{(0)}\mathcal{T}_{bl}^{(0)} = \frac{1}{\rho(\sigma)}\Delta_{\mathfrak{g}}(T)1\!\!1 \neq 0, \qquad \Delta_{\mathfrak{g}}(T)\delta_\alpha^\beta = \frac{(\eta^{ab}T_aT_b)_\alpha{}^\beta}{2k+Q_{\mathfrak{g}}} \tag{9.16b}$$

$$2\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{raj;-r,bl}(\sigma)\frac{\bar{r}}{\rho(\sigma)}\mathcal{T}_{aj}^{(r)}\mathcal{T}_{bl}^{(-r)} = \mathcal{L}_{\hat{\mathfrak{g}}(\sigma)}^{raj;-r,bl}(\sigma)\mathcal{T}_{aj}^{(r)}\mathcal{T}_{bl}^{(-r)}(1-\delta_{\bar{n}(r),0}) = \Delta_{\mathfrak{g}}(T)\frac{\rho(\sigma)-1}{\rho(\sigma)}1\!\!1\,, \tag{9.16c}$$

where $\alpha, \beta = 1 \ldots \dim T$ and $\Delta_{\mathfrak{g}}(T)$ is the conformal weight of each irrep T of each simple $\mathfrak{g}$. Following the more general discussion of subsection 9.1, we verify that the one-point correlators of the WZW cyclic permutation orbifolds vanish $\langle \hat{g}(\mathcal{T}, z, \sigma)\rangle_\sigma = 0$, $\sigma = 0, \ldots, \lambda - 1$ for non-trivial $\mathcal{T}$ because the quadratic structure (9.16b) is non-zero.

Using (8.5), (9.15) and (9.16), a more explicit form of the twisted left mover KZ equations for $A_g(\mathbb{Z}_\lambda)/\mathbb{Z}_\lambda$ is obtained

$$\partial_\mu \hat{A}_+(\sigma) = \hat{A}_+(\sigma)\hat{W}_\mu(\sigma)\,, \; \hat{A}_+(\sigma) \equiv \hat{A}_+(\mathcal{T}, z, \sigma)\,, \; \sigma = 0, \ldots, \lambda - 1 \tag{9.17a}$$

$$\hat{W}_\mu(\sigma) = \frac{2}{\rho(\sigma)}\frac{1}{2k+Q_{\mathfrak{g}}}\sum_{\nu \neq \mu}\eta^{ab}\frac{T_a^{(\nu)}T_b^{(\mu)}}{z_{\mu\nu}} \times$$

$$\times \sum_{r=0}^{\rho(\sigma)-1}\sum_{j=0}^{\frac{\lambda}{\rho(\sigma)}-1}\left(\frac{z_\nu}{z_\mu}\right)^{\frac{r}{\rho(\sigma)}} t_{rj}^{(\nu)}t_{-rj}^{(\mu)} - \frac{\Delta_{\mathfrak{g}}(T^{(\mu)})}{z_\mu}\frac{\rho(\sigma)-1}{\rho(\sigma)} \tag{9.17b}$$

$$\hat{A}_+(\sigma)\hat{Q}_{aj}(\sigma) = 0, \qquad a = 1 \ldots \dim \mathfrak{g}, \qquad j = 0, \ldots, \tfrac{\lambda}{\rho(\sigma)} - 1 \tag{9.17c}$$

$$\hat{Q}_{aj}(\sigma) = \sum_{\mu=1}^{N} T_a^{(\mu)}t_{0j}^{(\mu)}, \qquad [\hat{Q}_{aj}(\sigma), \hat{Q}_{bl}(\sigma)] = i\delta_{jl}f_{ab}{}^c\hat{Q}_{cj}(\sigma)\,, \tag{9.17d}$$

where we have included the global Ward identities of sector σ in (9.17c).

In this case, we verify the abelian flatness (8.7) of $\hat{W}_\mu(\sigma)$ as follows. By direct computation using (9.17b) and

$$[T_a^{(\mu)}, T_b^{(\nu)}] = [t_{rj}^{(\mu)}, t_{sj}^{(\nu)}] = 0 \quad \text{when } \nu \neq \mu \tag{9.18}$$

we find the proportionality

$$[\hat{W}_\mu(\sigma), \hat{W}_\nu(\sigma)] \sim \sum_{\epsilon \neq \mu,\nu}\sum_{r,s=0}^{\rho(\sigma)-1}\sum_{j=0}^{\frac{\lambda}{\rho(\sigma)}-1}\left\{ t_{-r,j}^{(\mu)}t_{r-s,j}^{(\nu)}t_{sj}^{(\epsilon)}\left(\frac{z_\nu}{z_\mu}\right)^{\frac{r}{\rho(\sigma)}}\left(\frac{z_\epsilon}{z_\nu}\right)^{\frac{s}{\rho(\sigma)}}\frac{1}{z_{\mu\nu}z_{\nu\epsilon}} + \right.$$

$$+t^{(\mu)}_{s-r,j}t^{(\nu)}_{-s,j}t^{(\epsilon)}_{rj}\left(\frac{z_\epsilon}{z_\mu}\right)^{\frac{r}{\rho(\sigma)}}\left(\frac{z_\mu}{z_\nu}\right)^{\frac{s}{\rho(\sigma)}}\frac{1}{z_{\mu\epsilon}z_{\nu\mu}}- \tag{9.19}$$
$$\left.-t^{(\mu)}_{-r,j}t^{(\nu)}_{-s,j}t^{(\epsilon)}_{r+s,j}\left(\frac{z_\epsilon}{z_\mu}\right)^{\frac{r}{\rho(\sigma)}}\left(\frac{z_\epsilon}{z_\nu}\right)^{\frac{s}{\rho(\sigma)}}\frac{1}{z_{\mu\epsilon}z_{\nu\epsilon}}\right\}if^{abc}T^{(\mu)}_aT^{(\nu)}_bT^{(\epsilon)}_c\,.$$

To check that this commutator vanishes

$$[\hat{W}_\mu(\sigma),\hat{W}_\nu(\sigma)]=0 \tag{9.20}$$

one needs only the identities

$$t^{(\mu)}_{r\pm\rho(\sigma),j}=t^{(\mu)}_{rj}\,,\qquad \frac{1}{z_{\mu\epsilon}z_{\nu\epsilon}}=\frac{1}{z_{\mu\nu}z_{\nu\epsilon}}+\frac{1}{z_{\mu\epsilon}z_{\nu\mu}} \tag{9.21}$$

and then one finds for each j that each distinct configuration $t^{(\mu)}_{pj}t^{(\nu)}_{qj}t^{(\epsilon)}_{rj}$ in (9.19) is multiplied by a sum of three terms which cancel. To see this analytically, the reader may find helpful the following intermediate relations

$$\sum_{s=0}^{\lambda-1}t^{(\nu)}_{r-s,j}t^{(\epsilon)}_{sj}y^{s/\lambda}=\sum_{s=0}^{\lambda-1}t^{(\nu)}_{-s,j}t^{(\epsilon)}_{r+s,j}y^{(r+s)/\lambda}+\frac{1-y}{y}\sum_{s=\lambda-r}^{\lambda-1}t^{(\nu)}_{-s,j}t^{(\epsilon)}_{r+s,j}y^{(r+s)/\lambda}\,,\ \forall\, y \tag{9.22a}$$
$$\sum_{r=0}^{\lambda-1}t^{(\mu)}_{s-r,j}t^{(\epsilon)}_{rj}x^{r/\lambda}=\sum_{s=0}^{\lambda-1}t^{(\mu)}_{-r,j}t^{(\epsilon)}_{r+s,j}x^{(r+s)/\lambda}+\frac{1-x}{x}\sum_{r=\lambda-s}^{\lambda-1}t^{(\mu)}_{-r,j}t^{(\epsilon)}_{r+s,j}x^{(r+s)/\lambda}\,,\ \forall\, x \tag{9.22b}$$
$$\sum_{s=0}^{\lambda-1}\sum_{r=\lambda-s}^{\lambda-1}f_{rsj}=\sum_{r=0}^{\lambda-1}\sum_{s=\lambda-r}^{\lambda-1}f_{rsj},\qquad \forall\, f\,, \tag{9.22c}$$

where (9.22a) and (9.22b) are consequences of the t periodicity in (9.21).

We continue with $A_g(\mathbb{Z}_\lambda)/\mathbb{Z}_\lambda$ in the simple case λ=prime, where we have for the twisted sectors $\sigma\neq 0$

$$\rho(\sigma)=\lambda\,,\quad \bar{r},\bar{s},\bar{t}=0,\dots,\lambda-1\,,\quad j,l,m=0,\qquad t_{rj}\to t_r\,,\quad \sigma=1,\dots,\lambda-1$$
$$\mathcal{T}^{(r)}_a=T_at_r,\qquad t_rt_s=t_{r+s},\qquad t_0=1,\qquad (t_r)_s{}^t=\delta_{r+s-t,\ \mathrm{mod}\lambda}\,. \tag{9.23a}$$

In this case, we consider the twisted two-point left mover correlators and their global Ward identities

$$\hat{A}_+(1,2)\equiv\langle\hat{g}_+(\mathcal{T}^{(1)},z_1,\sigma)\hat{g}_+(\mathcal{T}^{(2)},z_2,\sigma)\rangle_\sigma\,,\qquad \hat{A}_+(1,2)(T^{(1)}_a+T^{(2)}_a)=0\,,$$
$$a=1\dots\dim\mathfrak{g} \tag{9.24}$$

which are a special case of the general discussion in subsection 9.2. The global identities imply that $\hat{A}_+(1,2)=0$ unless $\Delta_{\mathfrak{g}}(T^{(1)})=\Delta_{\mathfrak{g}}(T^{(2)})$ as in the untwisted case.

Then using the global Ward identities again, the twisted KZ connections of the two-point correlators may be written

$$\hat{W}_1 \simeq -\Delta_{\mathfrak{g}}(T^{(1)}) \left[\frac{2}{\lambda z_{12}} \left(1 + \sum_{r=1}^{\lambda-1} \left(\frac{z_2}{z_1} \right)^{\frac{r}{\lambda}} t_r^{(2)} t_{-r}^{(1)} \right) + \frac{\lambda - 1}{\lambda z_1} \right] \tag{9.25a}$$

$$\hat{W}_2 \simeq -\Delta_{\mathfrak{g}}(T^{(1)}) \left[\frac{2}{\lambda z_{21}} \left(1 + \sum_{r=1}^{\lambda-1} \left(\frac{z_1}{z_2} \right)^{1-\frac{r}{\lambda}} t_r^{(2)} t_{-r}^{(1)} \right) + \frac{\lambda - 1}{\lambda z_2} \right] \tag{9.25b}$$

$$[t_r^{(2)} t_{-r}^{(1)}, t_s^{(2)} t_{-s}^{(1)}] = 0 \, , \, \forall \, r, s \, . \tag{9.25c}$$

Integrating, we obtain the twisted two-point correlators[9] of twisted sector $\sigma = 1, \ldots, \lambda - 1$

$$\hat{A}_+(1,2) = C_+(\mathcal{T}) \left((z_{12})^{\frac{2}{\lambda}} (z_1 z_2)^{\frac{\lambda-1}{\lambda}} \prod_{r=1}^{\lambda-1} e^{\frac{2}{\lambda} I_{r/\lambda}(\frac{z_1}{z_2}) t_r^{(2)} t_{-r}^{(1)}} \right)^{-\Delta_{\mathfrak{g}}(T^{(1)})} , \, \lambda = \text{prime} \tag{9.26a}$$

$$I_{r/\lambda}(y) = \int^y \frac{dx}{x-1} x^{-r/\lambda}, \qquad C_+(\mathcal{T})(T_a^{(1)} + T_a^{(2)}) = 0 \tag{9.26b}$$

$$\hat{A}_+(1,2) \big(\overleftarrow{\partial_1} z_1 + \overleftarrow{\partial_2} z_2 + 2\Delta_{\mathfrak{g}}(T^{(1)}) \big) = 0 \, , \tag{9.26c}$$

where the integrals $I_{r/\lambda}$ are evaluated in appendix B. Because $I_{r/\lambda}(z_1/z_2)$ is scale invariant, this result also satisfies the $L_\sigma(0)$ Ward identity in (9.26c). Note that the result (9.26) is independent of the sector σ. This can be traced back to the fact that, for prime λ, the twisted current algebra (7.14) of each twisted sector reduces to a left and right mover copy of the same simple [33] orbifold affine algebra at level $\hat{k} = \lambda k$.

For $H = \mathbb{Z}_2$, the result (9.26) reduces to

$$\hat{A}_+(1,2) = C_+(\mathcal{T}) \left(\frac{1}{z_{12}} \frac{1}{\sqrt{z_1 z_2}} \left(\frac{\sqrt{\frac{z_1}{z_2}} + 1}{\sqrt{\frac{z_1}{z_2}} - 1} \right)^{t_1^{(2)} t_1^{(1)}} \right)^{\Delta_{\mathfrak{g}}(T^{(1)})} ,$$

$$C_+(\mathcal{T})(T_a^{(1)} + T_a^{(2)}) = 0 \tag{9.27a}$$

$$t_0^{(\mu)} = 1, \qquad t_1^{(\mu)} = \tau_1^{(\mu)}, \qquad \mu = 1, 2, \qquad t_1^{(2)} t_1^{(1)} \in \{0 \text{ or } 1\} \tag{9.27b}$$

for the single twisted sector $\sigma = 1$. In this result, τ_1 is the first Pauli matrix.

For the case λ=prime and $\sigma \neq 0$, we have also worked out the right mover two-point correlators and the full nonchiral product

$$\hat{A}(1,2) = \hat{A}_-(1,2) \hat{A}_+(1,2) =$$

$$= C_- C_+ \left(|z_{12}|^{\frac{4}{\lambda}} (|z_1||z_2|)^{\frac{2(\lambda-1)}{\lambda}} \right)^{-\Delta_{\mathfrak{g}}(T^{(1)})} e^{-\frac{2}{\lambda} \Delta_{\mathfrak{g}}(T^{(1)}) F(1,2)} \tag{9.28a}$$

[9]With (7.13), (9.16b) and the global Ward identity, the result (9.26) can also be obtained as a special case of the more general two-point correlator in (9.12).

$$F(1,2) \equiv \sum_{r=1}^{\lambda-1} [I_{\frac{r}{\lambda}}(\bar{y}, y_0) t_r^{(1)} t_{-r}^{(2)} + I_{\frac{r}{\lambda}}(y, y_0) t_r^{(2)} t_{-r}^{(1)}], \qquad y = \tfrac{z_1}{z_2} \tag{9.28b}$$

$$I_{\frac{\bar{n}(r)}{\rho(\sigma)}}(y, y_0) = \int_{y_0}^{y} \frac{dx}{x-1}\, x^{-\frac{\bar{n}(r)}{\rho(\sigma)}} = I_{\frac{\rho(\sigma)-\bar{n}(r)}{\rho(\sigma)}}(y^{-1}, y_0^{-1}) \tag{9.28c}$$

$$C_+(T_a^{(1)} + T_a^{(2)}) = (T_a^{(1)} + T_a^{(2)})C_- = 0\,, \tag{9.28d}$$

where y_0 is a fixed reference point for the integrals $I_{n/\rho}$ (see also appendix B). As expected, these correlators are $1 \leftrightarrow 2$ symmetric $\hat{A}(2,1) = \hat{A}(1,2)$. To see this, we note that $\Delta_{\mathfrak{g}}(T^{(1)}) = \Delta_{\mathfrak{g}}(T^{(2)})$ and

$$F(2,1) = \sum_{r=1}^{\lambda-1} \left[I_{\frac{r}{\lambda}}(\bar{y}^{-1}, y_0) t_r^{(2)} t_{-r}^{(1)} + I_{\frac{r}{\lambda}}(y^{-1}, y_0) t_r^{(1)} t_{-r}^{(2)} \right] \tag{9.29a}$$

$$= \sum_{r=1}^{\lambda-1} \left[I_{\frac{\lambda-r}{\lambda}}(\bar{y}, y_0^{-1}) t_{\lambda-r}^{(1)} t_{r-\lambda}^{(2)} + I_{\frac{\lambda-r}{\lambda}}(y, y_0^{-1}) t_{\lambda-r}^{(2)} t_{r-\lambda}^{(1)} \right] \tag{9.29b}$$

$$= \sum_{r=1}^{\lambda-1} \left[I_{\frac{r}{\lambda}}(\bar{y}, y_0^{-1}) t_r^{(1)} t_{-r}^{(2)} + I_{\frac{r}{\lambda}}(y, y_0^{-1}) t_r^{(2)} t_{-r}^{(1)} \right] = F(1,2) + \Delta \tag{9.29c}$$

$$\Delta = \sum_{r=1}^{\lambda-1} I_{\frac{r}{\lambda}}(y_0, y_0^{-1}) [t_r^{(1)} t_{-r}^{(2)} + t_r^{(2)} t_{-r}^{(1)}] = 0\,. \tag{9.29d}$$

Here we have used the symmetry relation (9.28c) and t periodicity to obtain (9.29b), and a relabelling to obtain (9.29c). The same relations are used in the rearrangement

$$\sum_{r=1}^{\lambda-1} I_{\frac{r}{\lambda}}(y_0, y_0^{-1}) t_r^{(1)} t_{-r}^{(2)} = \sum_{r=1}^{\lambda-1} I_{\frac{r}{\lambda}}(y_0^{-1}, y_0) t_r^{(2)} t_{-r}^{(1)} = -\sum_{r=1}^{\lambda-1} I_{\frac{r}{\lambda}}(y_0, y_0^{-1}) t_r^{(2)} t_{-r}^{(1)} \tag{9.30}$$

which shows that $\Delta = 0$.

10. The inner-automorphic WZW orbifolds

As our final large example, we work out the inner-automorphic WZW orbifolds [32]

$$\frac{A_g(H(d))}{H(d)}, \qquad H(d) \subset \mathrm{Lie}\, G \subset \mathrm{Aut}(g) \tag{10.1}$$

in further detail, where Lie G is the Lie group whose Lie algebra is compact simple Lie g. For these orbifolds, we concentrate on the rectification problem and the twisted KZ systems which describe the twisted left and right mover correlators.

10.1 Basic properties and rectification

Let $h_\sigma \in H(d) \subset \mathrm{Lie}\, G \subset \mathrm{Aut}(g)$ be an element of any group $H(d)$ of inner automorphisms of simple g. Following ref. [32], we know the matrix action $w(h_\sigma)$

$$w(h_\sigma)_\alpha{}^\beta = \delta_\alpha{}^\beta e^{2\pi i \sigma \alpha \cdot d}, \quad w(h_\sigma)_A{}^B = \delta_A{}^B, \quad w(h_\sigma)_\alpha{}^A = w(h_\sigma)_A{}^\alpha = 0 \tag{10.2a}$$

$$\alpha \in \Delta(g), \quad A = 1 \ldots \text{rank } g, \quad \sigma = 0, \ldots, \rho(1) - 1, \quad N_c = \rho(1) \tag{10.2b}$$

of h_σ on the untwisted currents in the Cartan-Weyl basis. This is the action of h_σ in the adjoint $(T_a^{adj})_b{}^c = -if_{ab}{}^c$. Refs. [24] and [32] discuss the conditions on the vector d such that $H(d)$ is a group of finite order.

Then the matrix action $W(h_\sigma; T)$ of h_σ in representation T must solve the linkage relation (3.5a) in the form

$$W^\dagger(h_\sigma; T) T_\alpha W(h_\sigma; T) = e^{2\pi i \sigma \alpha \cdot d} T_\alpha \,, \quad W^\dagger(h_\sigma; T) T_A W(h_\sigma; T) = T_A \,. \tag{10.3}$$

To solve these equations, it is convenient to introduce the weight basis of matrix representation T

$$(T_A)_\lambda{}^{\lambda'} \equiv \langle T, \lambda | H_A(0) | T, \lambda' \rangle = \lambda_A \delta_{\lambda,\lambda'} \,, \; (T_\alpha)_\lambda{}^{\lambda'} \equiv \langle T, \lambda | E_\alpha(0) | T, \lambda' \rangle \propto \delta_{\lambda,\lambda'+\alpha} \tag{10.4a}$$

$$[T_A, T_B] = 0, \qquad [T_A, T_\alpha] = \alpha_A T_\alpha, \tag{10.4b}$$

$$[T_\alpha, T_\beta] = \begin{cases} N_\gamma(\alpha,\beta) T_\gamma & \text{if } \alpha + \beta = \gamma \\ \alpha \cdot T & \text{if } \alpha + \beta = 0 \\ 0 & \text{otherwise} \end{cases} \tag{10.4c}$$

$$N_\gamma(\alpha, \beta) = -N_\gamma(\beta, \alpha) = N_{-\beta}(-\gamma, \alpha) = -N_{-\gamma}(-\alpha, -\beta) \,, \tag{10.4d}$$

where $\{\lambda\}$ are the weights of the representation. The solution of the linkage relation

$$W(h_\sigma; T)_\lambda{}^{\lambda'} = \delta_\lambda{}^{\lambda'} e^{-2\pi i \sigma \lambda \cdot d}, \quad W^\dagger(h_\sigma; T)_\lambda{}^{\lambda'} = \delta_\lambda{}^{\lambda'} e^{+2\pi i \sigma \lambda \cdot d} \tag{10.5}$$

is obtained immediately in this basis. Note that the adjoint matrices of the weight basis are similarity transformations of $(T_a^{\text{adj}})_b{}^c = -if_{ab}{}^c$, so that W does not reduce to w in this case.

We discuss the orders $\rho(\sigma)$ and $R(T, \sigma)$ of $w(h_\sigma)$ and $W(h_\sigma; T)$, which are the smallest integers satisfying

$$e^{2\pi i \sigma \alpha \cdot d \rho(\sigma)} = 1, \quad e^{-2\pi i \sigma \lambda \cdot d R(T,\sigma)} = 1 \,. \tag{10.6}$$

These conditions imply that

$$\begin{cases} R(T,\sigma) = \rho(\sigma) & \text{when } T \text{ is in the conjugacy class of} \\ & \;\text{the root lattice} \\ e^{-2\pi i \sigma \lambda_{\text{min}} \cdot d R(T,\sigma)} = 1 & \text{when } T \text{ is in the conjugacy class of} \\ & \;\lambda_{\text{min}} \text{ plus the root lattice} \end{cases} \tag{10.7}$$

where $\{\lambda_{\text{min}}\}$ are the minimal weights of simple g, given in ref. [40]. As the simplest example, we find

$$R(T, \sigma) = \begin{cases} \rho(\sigma) & \text{for integral spin } j \\ 2\rho(\sigma) & \text{for half-integral spin } j \end{cases} \tag{10.8}$$

for $g = \text{SU}(2)$. In what follows, we revert to our abbreviation $R(\sigma) \equiv R(T, \sigma)$.

In (10.2) and (10.5), the actions $w(h_\sigma)$ and $W(h_\sigma;T)$ are diagonal, so the solutions of the H-eigenvalue problem and the extended H-eigenvalue problem can be taken as:

$$\begin{aligned} &U^\dagger(h_\sigma) = \chi(\sigma) = 1\,, \quad U^\dagger(T,h_\sigma) = \mathbb{1}\,, \quad \mathcal{G}(\sigma) = G\,, \quad \mathcal{F}(\sigma) = f\,, \\ &(T,\sigma) = T \qquad\qquad (10.9a) \\ &n_A = 0, \quad \frac{n_\alpha(r)}{\rho(\sigma)} = -\sigma\alpha\cdot d, \quad \frac{N(r)}{R(\sigma)} = \sigma\lambda\cdot d\,, \\ &\mathcal{M}(\mathcal{T},\sigma) = M(k,T) = \frac{k}{y(T)}\mathbb{1}\,. \qquad\qquad (10.9b) \end{aligned}$$

This tells us that all the selection rules discussed above are satisfied naturally by the untwisted objects in the Cartan-Weyl basis and weight basis. For example, the selection rule (4.20) for $\mathcal{T}$ takes the form

$$e^{-2\pi i\sigma(\lambda-\lambda')\cdot d}(T_A)_\lambda{}^{\lambda'} = (T_A)_\lambda{}^{\lambda'}, \quad e^{-2\pi i\sigma(\lambda-\lambda')\cdot d}(T_\alpha)_\lambda{}^{\lambda'} = e^{-2\pi i\sigma\alpha\cdot d}(T_\alpha)_\lambda{}^{\lambda'} \qquad (10.10)$$

which is easily checked from (10.4a).

The inner-automorphic data in (10.9) gives the monodromies and mode expansions of the twisted currents of $A_g(H(d))/H(d)$

$$\begin{aligned} \hat{H}_A(ze^{2\pi i}) &= \hat{H}_A(z)\,, \qquad \hat{E}_\alpha(ze^{2\pi i}) = e^{2\pi i\sigma\alpha\cdot d}\hat{E}_\alpha(z) && (10.11a) \\ \hat{\bar{H}}_A(\bar{z}e^{-2\pi i}) &= \hat{\bar{H}}_A(\bar{z})\,, \qquad \hat{\bar{E}}_\alpha(\bar{z}e^{-2\pi i}) = e^{2\pi i\sigma\alpha\cdot d}\hat{\bar{E}}_\alpha(\bar{z}) && (10.11b) \\ \hat{H}_A(z) &= \sum_{m\in\mathbb{Z}} \hat{H}_A(m) z^{-m-1}\,,\ \hat{E}_\alpha(z) = \sum_{m\in\mathbb{Z}} \hat{E}_\alpha(m-\sigma\alpha\cdot d) z^{-(m-\sigma\alpha\cdot d)-1} && (10.11c) \\ \hat{\bar{H}}_A(\bar{z}) &= \sum_{m\in\mathbb{Z}} \hat{\bar{H}}_A(m)\bar{z}^{m-1}\,,\ \hat{\bar{E}}_\alpha(\bar{z}) = \sum_{m\in\mathbb{Z}} \hat{\bar{E}}_\alpha(m-\sigma\alpha\cdot d)\bar{z}^{(m-\sigma\alpha\cdot d)-1} && (10.11d) \end{aligned}$$

as a special case of (5.14). Then one finds that the twisted left mover current algebra is the standard [18, 41, 24, 42, 32] inner-automorphically twisted affine Lie algebra

$$\begin{aligned} &\hat{\mathfrak{g}}(h_\sigma) = \hat{\mathfrak{g}}(H(d)\subset \text{Lie}\,G \subset \text{Aut}(g);\sigma), \quad \sigma = 0,\ldots,\rho(1)-1 && (10.12a) \\ &[\hat{H}_A(m),\hat{H}_B(n)] = km\delta_{AB}\delta_{m+n,0}\,, \\ &[\hat{H}_A(m),\hat{E}_\alpha(n-\sigma\alpha\cdot d)] = \alpha_A\hat{E}_\alpha(m+n-\sigma\alpha\cdot d) && (10.12b) \\ &[\hat{E}_\alpha(m-\sigma\alpha\cdot d),\hat{E}_\beta(n-\sigma\beta\cdot d)] = \\ &\qquad = \begin{cases} N_\gamma(\alpha,\beta)\hat{E}_\gamma(m+n-\sigma\gamma\cdot d) & \text{if } \alpha+\beta=\gamma \\ \alpha\cdot\hat{H}(m+n) + k(m-\sigma\alpha\cdot d)\delta_{m+n,0} & \text{if } \alpha+\beta=0 \\ 0 & \text{otherwise} \end{cases} && (10.12c) \\ &A = 1\ldots\text{rank}\,g, \quad \alpha\in\Delta(g), \quad \sigma = 0,\ldots,\rho(1)-1 && (10.12d) \end{aligned}$$

and the commuting twisted right mover algebra $\hat{\mathfrak{g}}(h_\sigma^{-1})$ has the same form except for the sign change of the central terms.

In this case also we have been able to solve the rectification problem for the twisted right movers. Using (10.4d), one finds that the simplest rectification[10] for all σ is

$$\hat{\bar{H}}^{\sharp}_A(m) \equiv -\hat{\bar{H}}_A(-m), \quad \hat{\bar{E}}^{\sharp}_\alpha(m-\sigma\alpha\cdot d) \equiv -\hat{\bar{E}}_{-\alpha}(-m+\sigma\alpha\cdot d)\,. \tag{10.13}$$

The algebra of the rectified modes $\hat{\bar{J}}^{\sharp}$ is a copy of (10.12), so that $\hat{\mathfrak{g}}(h_\sigma^{-1}) \simeq \hat{\mathfrak{g}}(h_\sigma)$ for all sectors σ of the orbifolds $A_g(H(d))/H(d)$.

The stress tensors and twisted conformal weight matrices of the inner-automorphic WZW orbifolds are also very simple

$$\mathcal{L}_{\hat{\mathfrak{g}}(\sigma)} = L_g, \qquad \mathcal{D}_{\hat{\mathfrak{g}}(\sigma)}(\mathcal{T}) = D_g(T) = \Delta_g(T) \tag{10.14a}$$

$$\hat{T}_\sigma(z) = \frac{1}{2k+Q_g}\left(\sum_A :\hat{H}_A(z)\hat{H}_A(z): + \sum_\alpha :\hat{E}_\alpha(z)\hat{E}_{-\alpha}(z):\right) \tag{10.14b}$$

$$\hat{\bar{T}}_\sigma(\bar{z}) = \frac{1}{2k+Q_g}\left(\sum_A :\hat{\bar{H}}_A(\bar{z})\hat{\bar{H}}_A(\bar{z}): + \sum_\alpha :\hat{\bar{E}}_\alpha(\bar{z})\hat{\bar{E}}_{-\alpha}(\bar{z}):\right) \tag{10.14c}$$

because the eigenvalue matrices U and $U(T)$ are unity. Here $\Delta_g(T)$ is the conformal weight of (untwisted) irrep T of (untwisted) g.

Similarly, we obtain the commutators with the twisted affine primary fields

$$\hat{g}(T,\bar{z},z)_\lambda{}^{\lambda'} = (\hat{g}_-(T,\bar{z})\hat{g}_+(T,z))_\lambda{}^{\lambda'}\,, \quad \hat{g}_+(T,z) \equiv \hat{g}_+(T,z,\sigma)\,,$$
$$\hat{g}_-(T,\bar{z}) \equiv \hat{g}_-(T,\bar{z},\sigma) \tag{10.15a}$$

$$[\hat{H}_A(m), \hat{g}_+(T,z)] = \hat{g}_+(T,z)z^m T_A\,,$$
$$[\hat{E}_\alpha(m-\sigma\alpha\cdot d), \hat{g}_+(T,z)] = \hat{g}_+(T,z)z^{m-\sigma\alpha\cdot d}T_\alpha \tag{10.15b}$$

$$[\hat{\bar{H}}_A(m), \hat{g}_-(T,\bar{z})] = -\bar{z}^{-m}T_A\hat{g}_-(T,\bar{z})\,,$$
$$[\hat{\bar{E}}_\alpha(m-\sigma\alpha\cdot d), \hat{g}_-(T,\bar{z})] = -\bar{z}^{-(m-\sigma\alpha\cdot d)}T_\alpha\hat{g}_-(T,\bar{z}) \tag{10.15c}$$

$$[L_\sigma(m), \hat{g}_+(T,z)] = \hat{g}_+(\mathcal{T},z)(\overleftarrow{\partial}\, z + (m+1)\Delta_g(T))z^m \tag{10.15d}$$

$$[\bar{L}_\sigma(m), \hat{g}_-(T,\bar{z})] = \bar{z}^m(\bar{z}\bar{\partial} + (m+1)\Delta_g(T))\hat{g}_-(T,\bar{z}) \tag{10.15e}$$

because $\mathcal{T} = T$ and $\mathcal{D}_{\hat{\mathfrak{g}}(\sigma)}(\mathcal{T}) = \Delta_g(T)$.

10.2 The untwisted affine vacuum and another mode normal ordering

For inner-automorphic orbifolds the action of the general twisted currents on the true ground state $|0\rangle_\sigma$ is not known (see refs. [24] and [32]). Here we provisionally study the action of the twisted currents on the untwisted affine vacuum $|0\rangle$

$$0 = (\hat{H}_A(m) - \delta_{m,0}\sigma k d_A)|0\rangle = \hat{E}_\alpha(m-\sigma\alpha\cdot d)|0\rangle, \qquad m \geq 0 \tag{10.16a}$$

[10] The rectification in (10.13) corresponds to the choice $\theta(\sigma) = -1$ for all σ in eq. (5.33a), and $\theta(0) = -1$ differs by a Chevalley involution from the trivial rectification $\theta(0) = 1$ discussed in subsection 5.4 for the untwisted sector. Unfortunately, $\theta(0) = 1$ is not a rectification beyond $\sigma = 0$.

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$$
\begin{aligned}
0 &= \langle 0|(\hat{H}_A(m) - \delta_{m,0}\sigma k d_A) = \langle 0|\hat{E}_\alpha(m - \sigma\alpha\cdot d), \qquad m \le 0 && (10.16b)\\
0 &= (\hat{\bar{H}}_A(m) + \delta_{m,0}\sigma k d_A)|0\rangle = \hat{\bar{E}}_\alpha(m - \sigma\alpha\cdot d)|0\rangle, \qquad m \le 0 && (10.16c)\\
0 &= \langle 0|(\hat{\bar{H}}_A(m) + \delta_{m,0}\sigma k d_A) = \langle 0|\hat{\bar{E}}_\alpha(m - \sigma\alpha\cdot d), \qquad m \ge 0 && (10.16d)\\
A &= 1\ldots \operatorname{rank} g, \qquad \forall\alpha\in\Delta(g), \qquad \sigma = 0,\ldots,\rho(1)-1\ . && (10.16e)
\end{aligned}
$$

The left mover conditions (10.16a), (10.16b) were given in ref. [32] and the right mover conditions (10.16c), (10.16d) follow immediately from the right mover copies

$$
\begin{aligned}
0 &= (\hat{\bar{H}}^\sharp_A(m) - \delta_{m,0}\sigma k d_A)|0\rangle = \hat{\bar{E}}^\sharp_\alpha(m - \sigma\alpha\cdot d)|0\rangle, \qquad m \ge 0 && (10.17a)\\
0 &= \langle 0|(\hat{\bar{H}}^\sharp_A(m) - \delta_{m,0}\sigma k d_A) = \langle 0|\hat{\bar{E}}^\sharp_\alpha(m - \sigma\alpha\cdot d), \qquad m \le 0 && (10.17b)
\end{aligned}
$$

of the left mover conditions, where $\hat{\bar{J}}^\sharp$ are the rectified right mover currents in (10.13).

As an application of the commutators (10.15b), (10.15c) and the vacuum conditions (10.16), we give the global Ward identities for the residual symmetry associated to the Lie subalgebra (5.18) of the twisted current algebra:

$$
\begin{aligned}
&\hat{A}(\sigma) = \hat{A}_-(\sigma)\hat{A}_+(\sigma)\,,\\
&\hat{A}_\pm(\sigma) \equiv \langle 0|\hat{g}_\pm(1,\sigma)\cdots\hat{g}_\pm(N,\sigma)|0\rangle \equiv \langle \hat{g}_\pm(1,\sigma)\cdots\hat{g}_\pm(N,\sigma)\rangle && (10.18a)\\
&\langle[\hat{H}_A(0), \hat{g}_+(1,\sigma)\cdots\hat{g}_+(N,\sigma)]\rangle = 0 \ \Rightarrow\ \hat{A}_+(\sigma)Q_A = 0\,,\ A = 1\ldots\operatorname{rank} g && (10.18b)\\
&\langle[\hat{\bar{H}}_A(0), \hat{g}_-(1,\sigma)\cdots\hat{g}_-(N,\sigma)]\rangle = 0 \ \Rightarrow\ Q_A\hat{A}_-(\sigma) = 0\,,\ A = 1\ldots\operatorname{rank} g && (10.18c)\\
&Q_A \equiv \sum_{\mu=1}^{N} T_A^{(\mu)}, \qquad [Q_A, Q_B] = 0\ . && (10.18d)
\end{aligned}
$$

In the permutation orbifolds (see subsections 8.2 and 8.3), the subalgebra (5.18) provided all the residual symmetries of the twisted sectors, but, in this case, we also find further coordinate-dependent[11] Ward identities associated to the twisted root operators at $m = 0$:

$$
\langle[\hat{E}_\alpha(-\sigma\alpha\cdot d), \hat{g}_+(1,\sigma)\cdots\hat{g}_+(N,\sigma)]\rangle = 0 \ \Rightarrow\ \hat{A}_+(\sigma)Q_\alpha(z,\sigma) = 0\,,\ \forall\ \alpha\in\Delta(g) \tag{10.19a}
$$

$$
\langle[\hat{\bar{E}}_\alpha(-\sigma\alpha\cdot d)\hat{g}_-(1,\sigma)\cdots\hat{g}_-(N,\sigma)]\rangle = 0 \ \Rightarrow\ \bar{Q}_\alpha(\bar{z},\sigma)\hat{A}_-(\sigma) = 0\,,\ \forall\ \alpha\in\Delta(g)\,. \tag{10.19b}
$$

$$
\begin{aligned}
&Q_\alpha(z,\sigma) \equiv \sum_{\mu=1}^{N} z_\mu^{-\sigma\alpha\cdot d} T_\alpha^{(\mu)}, \qquad \bar{Q}_\alpha(\bar{z},\sigma) \equiv \sum_{\mu=1}^{N} \bar{z}_\mu^{\,\sigma\alpha\cdot d} T_\alpha^{(\mu)} && (10.19c)\\
&[Q_A, Q_\alpha(z,\sigma)] = \alpha_A Q_\alpha(z,\sigma), \qquad [Q_A, \bar{Q}_\alpha(\bar{z},\sigma)] = \alpha_A \bar{Q}_\alpha(\bar{z},\sigma) && (10.19d)\\
&[Q_\alpha(z), Q_\beta(z)] = \begin{cases} N_\gamma(\alpha,\beta)Q_\gamma(z) & \text{if } \alpha+\beta=\gamma\\ \alpha\cdot Q & \text{if } \alpha+\beta=0\\ 0 & \text{otherwise}\end{cases} && (10.19e)
\end{aligned}
$$

[11] Coordinate-dependent Ward identities have also been encountered in affine-Virasoro theory [43, 4].

$$
[\bar{Q}_\alpha(\bar{z}), \bar{Q}_\beta(\bar{z})] = \begin{cases} N_\gamma(\alpha,\beta)\bar{Q}_\gamma(\bar{z}) & \text{if } \alpha+\beta=\gamma \\ \alpha\cdot Q & \text{if } \alpha+\beta=0 \\ 0 & \text{otherwise .} \end{cases} \tag{10.19f}
$$

We continue with the M' and $\bar{M}'$ normal orderings of the twisted currents

$$
\begin{aligned}
&: \hat{J}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)})\hat{J}_{n(s)\nu}(n+\tfrac{n(s)}{\rho(\sigma)}) :_{M'} \equiv \\
&\quad \equiv \theta(m\geq 0)\ \hat{J}_{n(s)\nu}\left(n+\frac{n(s)}{\rho(\sigma)}\right)\hat{J}_{n(r)\mu}\left(m+\frac{n(r)}{\rho(\sigma)}\right)+ \\
&\qquad +\theta(m<0)\ \hat{J}_{n(r)\mu}\left(m+\frac{n(r)}{\rho(\sigma)}\right)\hat{J}_{n(s)\nu}\left(n+\frac{n(s)}{\rho(\sigma)}\right) \\
&: \hat{\bar{J}}_{n(r)\mu}(m+\tfrac{n(r)}{\rho(\sigma)})\hat{\bar{J}}_{n(s)\nu}\left(n+\frac{n(s)}{\rho(\sigma)}\right) :_{\bar{M}'} \equiv \\
&\quad \equiv \theta(m\leq 0)\ \hat{\bar{J}}_{n(s)\nu}\left(n+\frac{n(s)}{\rho(\sigma)}\right)\hat{\bar{J}}_{n(r)\mu}\left(m+\frac{n(r)}{\rho(\sigma)}\right)+ \\
&\qquad +\theta(m>0)\ \hat{\bar{J}}_{n(r)\mu}\left(m+\frac{n(r)}{\rho(\sigma)}\right)\hat{\bar{J}}_{n(s)\nu}\left(n+\frac{n(s)}{\rho(\sigma)}\right)
\end{aligned} \tag{10.20a}
$$

$$
\hat{J} = \{\hat{H}_A, \hat{E}_\alpha\}, \qquad \hat{\bar{J}} = \{\hat{\bar{H}}_A, \hat{\bar{E}}_\alpha\}, \qquad n_A = 0, \qquad \tfrac{n_\alpha(r)}{\rho(\sigma)} = -\sigma\alpha\cdot d \tag{10.20b}
$$

which are tailored for the untwisted affine vacuum $|0\rangle$. The M ordering in (10.20a) was studied in ref. [32].

Then we can obtain the relations between the various normal ordered products, and, using $\sum_\alpha \alpha_A\alpha_B = Q_g\delta_{AB}$, we obtain the mode ordered form of the left and right mover Virasoro generators

$$
\begin{aligned}
(2k+Q_g)L_\sigma(m) = \sum_{p\in\mathbb{Z}}\Bigg\{&: \sum_\alpha \hat{E}_\alpha(p-\sigma\alpha\cdot d)\hat{E}_{-\alpha}(m-p+\sigma\alpha\cdot d) :_{M'} + \\
&+\sum_A : \hat{H}_A(p)\hat{H}_A(m-p) :_{M'}\Bigg\} + Q_g\{\sigma d\cdot\hat{H}(m) - \delta_{m,0}\frac{k}{2}\sigma^2 d^2\Bigg\}
\end{aligned} \tag{10.21a}
$$

$$
\begin{aligned}
(2k+Q_g)\bar{L}_\sigma(m) = \sum_{p\in\mathbb{Z}}\Bigg\{&: \sum_\alpha \hat{\bar{E}}_\alpha(p-\sigma\alpha\cdot d)\hat{\bar{E}}_{-\alpha}(-m-p+\sigma\alpha\cdot d) :_{\bar{M}'} + \\
&+\sum_A : \hat{\bar{H}}_A(p)\hat{\bar{H}}_A(-m-p) :_{\bar{M}'}\Bigg\} - Q_g\{\sigma d\cdot\hat{\bar{H}}(-m) + \delta_{m,0}\frac{k}{2}\sigma^2 d^2\Bigg\}.
\end{aligned} \tag{10.21b}
$$

The left mover generators (10.21a) were given in ref. [32].

Then using the vacuum conditions (10.16) we verify that the untwisted affine vacuum is indeed primary under Vir⊕Vir

$$
(L_\sigma(m\geq 0) - \delta_{m,0}\hat{\Delta}_0(\sigma))|0\rangle = (\bar{L}_\sigma(m\geq 0) - \delta_{m,0}\hat{\bar{\Delta}}_0(\sigma))|0\rangle = 0\,,
$$

$$\hat{\Delta}_0(\sigma) = \hat{\bar{\Delta}}_0(\sigma) = \frac{k}{2}\sigma^2 d^2 \tag{10.22}$$

with the same left and right mover conformal weights. The left mover result in (10.22) is of course very old [24]. Consistent with (10.22), it is not difficult to verify that the set of right mover Virasoro generators in (10.21b) can be expressed as a copy

$$(2k+Q_g)\bar{L}_\sigma(m) = \sum_{p\in\mathbb{Z}}\Big\{ : \sum_\alpha \hat{\bar{E}}^\sharp_\alpha(p-\sigma\alpha\cdot d)\hat{\bar{E}}^\sharp_{-\alpha}(m-p+\sigma\alpha\cdot d) :_{M'} + \tag{10.23}$$

$$+\sum_A : \hat{\bar{H}}^\sharp_A(p)\hat{\bar{H}}^\sharp_A(m-p) :_{M'}\} + Q_g\{\sigma d\cdot \hat{\bar{H}}^\sharp(m) - \delta_{m,0}\frac{k}{2}\sigma^2 d^2\Big\}$$

of the left mover Virasoro generators in (10.21a). Here, $\{\hat{\bar{J}}^\sharp\}$ are the rectified right mover current modes in (10.13). As noted in subsection 7.5, this situation is expected when the twisted right mover currents are rectifiable.

We will also need the corresponding M' and $\bar{M}'$ normal orderings of the twisted currents with the twisted affine primary fields

$$: \hat{H}_A(m)\hat{g}_+(T,z) :_{M'} \equiv \theta(m\geq 0)\hat{g}_+(T,z)\hat{H}_A(m) + \theta(m<0)\hat{H}_A(m)\hat{g}_+(T,z) \tag{10.24a}$$

$$: \hat{E}_\alpha(m-\sigma\alpha\cdot d)\hat{g}_+(T,z) :_{M'} \tag{10.24b}$$

$$\equiv \theta(m\geq 0)\hat{g}_+(T,z)\hat{E}_\alpha(m-\sigma\alpha\cdot d) + \theta(m<0)\hat{E}_\alpha(m-\sigma\alpha\cdot d)\hat{g}_+(T,z)$$

$$: \hat{\bar{H}}_A(m)\hat{g}_-(T,\bar{z}) :_{\bar{M}'} \equiv \theta(m\leq 0)\hat{g}_-(T,\bar{z})\hat{\bar{H}}_A(m) + \theta(m>0)\hat{\bar{H}}_A(m)\hat{g}_-(T,\bar{z}) \tag{10.24c}$$

$$: \hat{\bar{E}}_\alpha(m-\sigma\alpha\cdot d)\hat{g}_-(T,\bar{z}) :_{\bar{M}'} \tag{10.24d}$$

$$\equiv \theta(m\leq 0)\hat{g}_-(T,\bar{z})\hat{\bar{E}}_\alpha(m-\sigma\alpha\cdot d) + \theta(m>0)\hat{\bar{E}}_\alpha(m-\sigma\alpha\cdot d)\hat{g}_-(T,\bar{z})\,. \tag{10.24e}$$

From these definitions and the commutators in (10.15b), (10.15c) one computes as above the exact operator products of the twisted currents with the twisted primary fields. This gives the relations

$$: \hat{H}_A(z)\hat{g}_+(T,z) :=: \hat{H}_A(z)\hat{g}_+(T,z) :_{M'}, \qquad : \hat{\bar{H}}_A(\bar{z})\hat{g}_-(T,\bar{z}) :=: \hat{\bar{H}}_A(\bar{z})\hat{g}_-(T,\bar{z}) :_{\bar{M}'} \tag{10.25a}$$

$$: \hat{E}_\alpha(z)\hat{g}_+(T,z) :=: \hat{E}_\alpha(z)\hat{g}_+(T,z) :_{M'} + \frac{\sigma\alpha\cdot d}{z}\hat{g}_+(T,z)T_\alpha \tag{10.25b}$$

$$: \hat{\bar{E}}_\alpha(\bar{z})\hat{g}_-(T,\bar{z}) :=: \hat{\bar{E}}_\alpha(\bar{z})\hat{g}_-(T,\bar{z}) :_{\bar{M}'} + \frac{\sigma\alpha\cdot d}{\bar{z}}T_\alpha\hat{g}_-(T,\bar{z}) \tag{10.25c}$$

among the normal ordered products.

10.3 Twisted vertex operator equations in mode-ordered form

The M' and $\bar{M}'$ ordered forms of the twisted vertex operator equations

$$\partial\hat{g}_+(T,z) = \frac{2}{2k+Q_g}\Big(\sum_A : \hat{H}_A(z)\hat{g}_+(T,z) :_{M'} T_A + \sum_\alpha : \hat{E}_\alpha(z)\hat{g}_+(T,z) :_{M'} T_{-\alpha} +$$

$$+\sum_\alpha \frac{\sigma\alpha\cdot d}{z}\hat{g}_+(T,z)T_\alpha T_{-\alpha}\Bigg) \tag{10.26a}$$

$$\bar\partial\hat{g}_-(T,\bar z) = -\frac{2}{2k+Q_g}\Bigg(\sum_A :\hat{\bar{H}}_A(\bar z)T_A\hat{g}_-(T,\bar z):_{\bar M'} + \sum_\alpha :\hat{\bar{E}}_\alpha(z)T_{-\alpha}\hat{g}_-(T,\bar z):_{\bar M'} +$$
$$+\sum_\alpha \frac{\sigma\alpha\cdot d}{\bar z}T_{-\alpha}T_\alpha\hat{g}_-(T,\bar z)\Bigg)\,, \tag{10.26b}$$

are obtained from (5.13) and (10.25). The explicit forms of these mode ordered products

$$:\hat{J}_a(z)\hat{g}_+(T,z):_{M'} = \hat{J}_a^-(z)\hat{g}_+(T,z) + \hat{g}_+(T,z)\hat{J}_a^+(z)\,,\; a=1\ldots\dim g \tag{10.27a}$$
$$\hat{H}_A^-(z) = \sum_{m\le -1}\hat{H}_A(m)z^{-m-1}\,,\; \hat{E}_\alpha^-(z) = \sum_{m\le -1}\hat{E}_\alpha(m-\sigma\alpha\cdot d)z^{-(m-\sigma\alpha\cdot d)-1} \tag{10.27b}$$
$$\hat{H}_A^+(z) = \sum_{m\ge 0}\hat{H}_A(m)z^{-m-1}\,,\; \hat{E}_\alpha^+(z) = \sum_{m\ge 0}\hat{E}_\alpha(m-\sigma\alpha\cdot d)z^{-(m-\sigma\alpha\cdot d)-1} \tag{10.27c}$$
$$:\hat{\bar{J}}_a(\bar z)\hat{g}_-(T,z):_{\bar M'} = \hat{\bar{J}}_a^+(\bar z)\hat{g}_-(T,\bar z) + \hat{g}_-(T,\bar z)\hat{\bar{J}}_a^-(\bar z)\,,\; a=1\ldots\dim g \tag{10.27d}$$
$$\hat{\bar{H}}_A^+(\bar z) = \sum_{m\ge 1}\hat{\bar{H}}_A(m)\bar z^{m-1}\,,\; \hat{\bar{E}}_\alpha^+(\bar z) = \sum_{m\ge 1}\hat{\bar{E}}_\alpha(m-\sigma\alpha\cdot d)\bar z^{(m-\sigma\alpha\cdot d)-1} \tag{10.27e}$$
$$\hat{\bar{H}}_A^-(\bar z) = \sum_{m\le 0}\hat{\bar{H}}_A(m)\bar z^{m-1}\,,\; \hat{\bar{E}}_\alpha^-(\bar z) = \sum_{m\le 0}\hat{\bar{E}}_\alpha(m-\sigma\alpha\cdot d)\bar z^{(m-\sigma\alpha\cdot d)-1} \tag{10.27f}$$

follow from eq. (10.24).

Using (10.27) and (10.16) we find the vacuum conditions

$$\langle 0|\hat{H}_A^-(z) = 0,\qquad \hat{H}_A^+(z)|0\rangle = \frac{1}{z}\sigma k d_A|0\rangle,\qquad \hat{E}_\alpha^+(z)|0\rangle = \langle 0|\hat{E}_\alpha^-(z) = 0 \tag{10.28a}$$
$$\langle :\hat{H}_A(z)\hat{g}_+(T,z):_{M'}\rangle = \frac{1}{z}\sigma k d_A\langle\hat{g}_+(T,z)\rangle,\qquad \langle :\hat{E}_\alpha(z)\hat{g}_+(T,z):_{M'}\rangle = 0 \tag{10.28b}$$
$$\langle 0|\hat{\bar{H}}_A^+(\bar z) = 0,\qquad \hat{\bar{H}}_A^-(\bar z)|0\rangle = -\frac{1}{\bar z}\sigma k d_A|0\rangle,\qquad \hat{\bar{E}}_\alpha^-(\bar z)|0\rangle = \langle 0|\hat{\bar{E}}_\alpha^+(\bar z) = 0 \tag{10.28c}$$
$$\langle :\hat{\bar{H}}_A(\bar z)\hat{g}_-(T,\bar z):_{\bar M'}\rangle = -\frac{1}{\bar z}\sigma k d_A\langle\hat{g}_-(T,\bar z)\rangle,\qquad \langle :\hat{\bar{E}}_\alpha(\bar z)\hat{g}_-(T,\bar z):_{\bar M'}\rangle = 0 \tag{10.28d}$$

and the commutators of the partial currents with the twisted affine primary fields are computed as above.

10.4 The twisted KZ equations of $A_g(H(d))/H(d)$

Using the vacuum conditions (10.28b) and (10.28c), the twisted vertex operator equation (10.26a) and the identity $\sum_\alpha \alpha\cdot d\, T_\alpha T_{-\alpha} = \frac{1}{2}Q_g\, d\cdot T$, one finds after some algebra the *twisted KZ equations*

$$\hat{A}_+(\sigma) \equiv \hat{A}_+(T,z,\sigma) = \langle\hat{g}_+(T^{(1)},z_1,\sigma)\cdots\hat{g}_+(T^{(N)},z_N,\sigma)\rangle \tag{10.29a}$$

$$\partial_\mu \hat{A}_+(\sigma) = \hat{A}_+(\sigma)\hat{W}_\mu(\sigma), \qquad \hat{W}_\mu(\sigma) \equiv \hat{W}_\mu(T,z,\sigma), \qquad \sigma = 0,\ldots,\rho(1)-1 \tag{10.29b}$$

$$\hat{W}_\mu(\sigma) = \frac{2}{2k+Q_g}\sum_{\nu\neq\mu}\frac{1}{z_{\mu\nu}}\left[\sum_A T_A^{(\nu)}T_A^{(\mu)} + \sum_\alpha \left(\frac{z_\nu}{z_\mu}\right)^{-\sigma\alpha\cdot d} T_\alpha^{(\nu)}T_{-\alpha}^{(\mu)}\right] + \frac{\sigma d\cdot T^{(\mu)}}{z_\mu} \tag{10.29c}$$

for each twisted left mover sector σ of all inner-automorphic WZW orbifolds. This twisted KZ connection is *non-abelian* flat

$$\begin{aligned}\partial_\mu \hat{W}_\nu(\sigma) - \partial_\nu \hat{W}_\mu(\sigma) &= -[\hat{W}_\mu(\sigma), \hat{W}_\nu(\sigma)] \\ &= -\frac{1}{z_\mu z_\nu}\frac{2}{2k+Q_g}\sum_\alpha \left(\frac{z_\mu}{z_\nu}\right)^{-\sigma\alpha\cdot d}\sigma\alpha\cdot d T_\alpha^{(\mu)}T_{-\alpha}^{(\nu)}.\end{aligned} \tag{10.30}$$

To our knowledge, all previous KZ-like connections in conformal field theory have been abelian flat. This remark includes the KZ-like connections recently obtained for open WZW strings [44]. We have also checked the consistency of the Ward identities (10.18b) and (10.19a) with the twisted KZ equations.

To obtain the corresponding results for the right mover sector of the inner-automorphic orbifolds, we begin with the twisted right mover vertex operator equation (10.26b). Then, using the vacuum conditions (10.28c), (10.28d), one finds the twisted right mover KZ equations

$$\hat{A}_-(\sigma) \equiv \langle \hat{g}_-(T^{(1)},\bar{z}_1,\sigma)\cdots\hat{g}_-(T^{(N)},\bar{z}_N,\sigma)\rangle, \qquad \sigma = 0,\ldots,\rho(1)-1 \tag{10.31a}$$

$$\bar{\partial}_\mu \hat{A}_-(\sigma) = \hat{\bar{W}}_\mu(\sigma)\hat{A}_-(\sigma) \tag{10.31b}$$

$$\hat{\bar{W}}_\mu(\sigma) = \frac{2}{2k+Q_g}\sum_{\nu\neq\mu}\frac{1}{\bar{z}_{\mu\nu}}\left[\sum_A T_A^{(\mu)}T_A^{(\nu)} + \sum_\alpha \left(\frac{\bar{z}_\nu}{\bar{z}_\mu}\right)^{-\sigma\alpha\cdot d} T_\alpha^{(\mu)}T_{-\alpha}^{(\nu)}\right] + \frac{\sigma d\cdot T^{(\mu)}}{\bar{z}_\mu} \tag{10.31c}$$

for each twisted right mover sector σ of the inner-automorphic WZW orbifolds. As in the case of the left movers, the twisted right mover connections $\hat{\bar{W}}_\mu$ are non-abelian flat.

10.5 Relation to the untwisted sector

Our results for the inner-automorphic WZW orbifolds are simpler than they appear. In particular, we are able to relate the twisted KZ systems (10.18), (10.19), (10.29) and (10.31) in twisted sector $\sigma \neq 0$ to the corresponding ordinary KZ systems [10] of the untwisted sector $\sigma = 0$. The relations are

$$\hat{A}_+(T,z,\sigma) = \hat{A}_+(T,z,0)U(T,z,\sigma), \qquad \hat{A}_-(T,\bar{z},\sigma) = \bar{U}(T,\bar{z},\sigma)\hat{A}_-(T,\bar{z},0) \tag{10.32a}$$

$$U(T,z,\sigma) \equiv \prod_{\mu=1}^{N} z_\mu{}^{\sigma d\cdot T^{(\mu)}}\,,\ \bar{U}(T,\bar{z},\sigma) \equiv \prod_{\mu=1}^{N} \bar{z}_\mu{}^{\sigma d\cdot T^{(\mu)}} \tag{10.32b}$$

$$\partial_\mu \hat{A}_+(T,z,0) = \hat{A}_+(T,z,0)\hat{W}_\mu(T,z,0), \qquad \bar{\partial}_\mu \hat{A}_-(T,\bar{z},0) = \hat{\bar{W}}_\mu(T,\bar{z},0)\hat{A}_-(T,\bar{z},0) \tag{10.32c}$$

$$\hat{W}_\mu(T,z,0) = \frac{2\eta^{ab}}{2k+Q_g}\sum_{\nu\neq\mu}\frac{T_a^{(\nu)}T_b^{(\mu)}}{z_{\mu\nu}}\,,\ \hat{\bar{W}}_\mu(T,\bar{z},0) = \frac{2\eta^{ab}}{2k+Q_g}\sum_{\nu\neq\mu}\frac{T_a^{(\mu)}T_b^{(\nu)}}{\bar{z}_{\mu\nu}} \tag{10.32d}$$

$$\hat{A}_+(T,z,0)Q_a(T,0) = Q_a(T,0)\hat{A}_-(T,\bar{z},0) = 0, \qquad Q_a(T,0) = \sum_{\mu=1}^{N} T_a^{(\mu)}$$
$$a = 1\ldots \dim g\,. \tag{10.32e}$$

In checking this equivalence one uses the identity

$$x^{\sigma d\cdot T}T_\alpha x^{-\sigma d\cdot T} = x^{\sigma\alpha\cdot d}T_\alpha, \qquad \forall\, x, \alpha \in \Delta(g) \tag{10.33}$$

and one also finds the relations

$$\hat{W}_\mu(T,z,\sigma) = U^{-1}(T,z,\sigma)(\hat{W}_\mu(T,z,0)+\partial_\mu)U(T,z,\sigma) \tag{10.34a}$$
$$\hat{\bar{W}}_\mu(T,\bar{z},\sigma) = \bar{U}(T,\bar{z},\sigma)(\hat{\bar{W}}_\mu(T,\bar{z},0)-\bar{\partial}_\mu)\bar{U}^{-1}(T,\bar{z},\sigma) \tag{10.34b}$$
$$U(T,z,\sigma)Q_\alpha(z,\sigma)U^{-1}(T,z,\sigma) = \bar{U}^{-1}(T,\bar{z},\sigma)\bar{Q}_\alpha(\bar{z},\sigma)\bar{U}(T,\bar{z},\sigma) = Q_\alpha(T,0) \tag{10.34c}$$

among the connections and generators of the Ward identities.

The equivalence relations (10.32) tell us that the twisted affine primary fields have the simple form

$$\hat{g}_+(T,z,\sigma) = \hat{g}_+(T,z,0)z^{\sigma d\cdot T}, \qquad \hat{g}_-(T,\bar{z},\sigma) = \bar{z}^{\sigma d\cdot T}\hat{g}_-(T,\bar{z},0) \tag{10.35}$$

in terms of the untwisted affine primary fields at $\sigma = 0$. As a consequence of (10.32) and (10.35), we also obtain the relations

$$\hat{A}(T,\bar{z},z,\sigma) = \bar{U}(T,\bar{z},\sigma)\hat{A}(T,\bar{z},z,0)U(T,z,\sigma)$$
$$\hat{g}(T,\bar{z},z,\sigma) = \bar{z}^{\sigma d\cdot T}\hat{g}(T,\bar{z},z,0)z^{\sigma d\cdot T} \tag{10.36}$$

for the full orbifold correlators $\hat{A} = \hat{A}_-\hat{A}_+$ and primary fields $\hat{g} = \hat{g}_-\hat{g}_+$.

As a check on these results, we may use the classical limit of (10.36) to compute the classical monodromies (see subsection 5.7) of the group orbifold elements in this case

$$\hat{g}(T,\bar{z}e^{-2\pi i},ze^{2\pi i},\sigma) = e^{-2\pi i\sigma d\cdot T}\hat{g}(T,\bar{z},z,\sigma)e^{2\pi i\sigma d\cdot T} \tag{10.37a}$$
$$\hat{g}(T,\bar{z}e^{-2\pi i},ze^{2\pi i},\sigma)_\lambda{}^{\lambda'} = e^{-2\pi i\sigma d\cdot(\lambda-\lambda')}\hat{g}(T,\bar{z},z,\sigma)_\lambda{}^{\lambda'}\,, \tag{10.37b}$$

where we have used (10.4a) and the fact that the classical group elements $\hat{g}(T,\bar{z},z,0)$ have trivial monodromy. Using the inner-automorphic data (10.9b), we see that the monodromies (10.37) are in agreement with the general formula (5.44) for the monodromies of the group orbifold elements.

We finally give the description of the left and right mover inner-automorphically twisted current algebra in terms of spectral flow. In our formulation the correct flows are

$$\hat{H}_A(m) = H_A(m) + \delta_{m,0} k\sigma d_A\,, \qquad \hat{E}_\alpha(m - \sigma\alpha\cdot d) = E_\alpha(m) \tag{10.38a}$$
$$\hat{\bar{H}}_A(m) = \bar{H}_A(-m) - \delta_{m,0} k\sigma d_A\,, \qquad \hat{\bar{E}}_\alpha(m - \sigma\alpha\cdot d) = \bar{E}_\alpha(-m)\,, \tag{10.38b}$$

where $\{J, \bar{J}\}$ are the ordinary untwisted modes of affine $(g \oplus g)$ (which satisfy (2.16c), (2.16d) and (2.16f)). The left mover flow in (10.38a) is well known [18, 41, 24, 42, 32], while the right mover flow in (10.38b) is obtained from (10.16c) and comparison of (2.16f) with (10.15c) at $\sigma = 0$. The right mover flow (10.38b) also gives the spectral flow

$$\hat{\bar{H}}^{\sharp}_A(m) = \bar{H}_A(m)' + \delta_{m,0} k\sigma d_A\,, \qquad \hat{\bar{E}}^{\sharp}_\alpha(m - \sigma\alpha\cdot d) = \bar{E}_\alpha(m)' \tag{10.39a}$$
$$\bar{H}_A(m)' \equiv -\bar{H}_A(m), \qquad \bar{E}_\alpha(m)' \equiv -\bar{E}_{-\alpha}(m) \tag{10.39b}$$

for the rectified right mover modes in (10.13). Relative to the left mover flow (10.38a), the extra Chevalley involution in (10.39b) is the same as that discussed near (10.13).

The spectral flows (10.38) and (10.39) are algebraic results which are independent of our provisional choice of the untwisted affine vacuum $|0\rangle$ for the twisted correlators. Moreover, given the flows (10.38) and (10.39), we find that the simple operator results (10.35) can be obtained directly by comparison of (2.16f) with (10.15b), (10.15c). It follows that the simple operator results (10.35) and (10.36) are also independent of our provisional choice of the untwisted affine vacuum.

Acknowledgments

For helpful discussions, we thank R. Dijkgraaf, J. Evslin, R. Littlejohn, C. Park, M. Porrati, C. Schweigert, H. Verlinde, J. Wang and N. Warner. For hospitality, MBH and NO thank the Niels Bohr Institute, and JdB and NO thank the University of California, Berkeley and Lawrence Berkeley National Laboratory. The work of JdB was supported in part by the National Science Foundation under Grant No PHY99-07949. The work of MBH was supported in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098 and in part by the National Science Foundation under grant PHY95-14797. The work of NO is supported in part by the stichting FOM and the European Commission RTN programme HPRN-CT-2000-00131.

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A. The general current-algebraic orbifold $A(H)/H$

The twisted left mover sectors of the general current-algebraic orbifold

$$\frac{A(H)}{H} \supset \frac{A_g(H)}{H}, \qquad H \subset \mathrm{Aut}(g)\,, \tag{A.1}$$

were studied in refs. [31, 32]. Here, we supplement that discussion by including the corresponding results for the twisted right mover sectors.

The stress tensors of the general current-algebraic orbifold are:

$$\hat{T}_\sigma(z) = \mathcal{L}^{n(r)\mu;-n(r),\nu}(\sigma) : \hat{J}_{n(r)\mu}(z)\hat{J}_{-n(r),\nu}(z) : \tag{A.2a}$$

$$= \sum_{r,\mu,\nu} \mathcal{L}^{n(r)\mu;-n(r),\nu}(\sigma)\Bigg\{ : \hat{J}_{n(r)\mu}(z)\hat{J}_{-n(r),\nu}(z) :_M - \\ - \frac{i\bar{n}(r)}{\rho(\sigma)}\mathcal{F}_{n(r)\mu;-n(r),\nu}{}^{0\delta}(\sigma)\frac{\hat{J}_{0\delta}(z)}{z} + \frac{1}{z^2}\frac{\bar{n}(r)}{2\rho(\sigma)}\left(1-\frac{\bar{n}(r)}{\rho(\sigma)}\right)\times \\ \times\mathcal{G}_{n(r)\mu;-n(r),\nu}(\sigma)\Bigg\} \tag{A.2b}$$

$$\hat{\bar{T}}_\sigma(\bar{z}) = \mathcal{L}^{n(r)\mu;-n(r),\nu}(\sigma) : \hat{\bar{J}}_{n(r)\mu}(\bar{z})\hat{\bar{J}}_{-n(r),\nu}(\bar{z}) : \tag{A.2c}$$

$$= \sum_{r,\mu,\nu} \mathcal{L}^{n(r)\mu;-n(r),\nu}(\sigma)\Bigg\{ : \hat{\bar{J}}_{n(r)\mu}(\bar{z})\hat{\bar{J}}_{-n(r),\nu}(\bar{z}) :_{\bar{M}} - \\ - \frac{i}{\bar{z}}\frac{\overline{-n(r)}}{\rho(\sigma)}\mathcal{F}_{n(r)\mu;-n(r),\nu}{}^{0\delta}(\sigma)\hat{\bar{J}}_{0\delta}(\bar{z}) + \frac{1}{\bar{z}^2}\frac{\overline{-n(r)}}{2\rho(\sigma)}\left(1-\frac{\overline{-n(r)}}{\rho(\sigma)}\right)\times \\ \times\mathcal{G}_{n(r)\mu;-n(r),\nu}(\sigma)\Bigg\} \tag{A.2d}$$

$$\mathcal{L}^{n(r)\mu;n(s)\nu}(\sigma) = \chi(\sigma)^{-1}_{n(r)\mu}\chi(\sigma)^{-1}_{n(s)\nu}L_H^{ab}U^\dagger(\sigma)_a{}^{n(r)\mu}U^\dagger(\sigma)_b{}^{n(s)\nu}\,,\ \sigma = 0,\ldots,N_c-1 \tag{A.2e}$$

$$= \delta_{n(r)+n(s),0 \bmod \rho(\sigma)}\mathcal{L}^{n(r)\mu;-n(r),\nu}(\sigma)\,. \tag{A.2f}$$

Here M and $\bar{M}$ normal ordering is defined in the text, L_H^{ab} is the H-invariant solution [13, 4] of the Virasoro master equation [5, 6, 4] corresponding to the H-invariant CFT $A(H)$, and $\{\mathcal{L}(\sigma)\}$ is the set of twisted inverse inertia tensors [31, 32] of the orbifold $A(H)/H$.

We also give the OPEs of the general stress tensors with the twisted currents

$$\hat{T}_\sigma(z)\hat{J}_{n(r)\mu}(w) = \mathcal{M}_{n(r)\mu}{}^{n(r)\mu}(\sigma)\left[\frac{1}{(z-w)^2}+\frac{\partial_w}{(z-w)}\right]\hat{J}_{n(r)\mu}(w) + \\ + \frac{\mathcal{N}_{n(r)\mu}{}^{n(s)\nu;n(r)-n(s),\delta}(\sigma) : \hat{J}_{n(s)\nu}(w)\hat{J}_{n(r)-n(s),\delta}(w) :}{z-w} + \mathcal{O}(z-w)^0 \tag{A.3a}$$

$$\hat{\bar{T}}_\sigma(\bar{z})\hat{\bar{J}}_{n(r)\mu}(\bar{w}) = \mathcal{M}_{n(r)\mu}{}^{n(r)\mu}(\sigma)[\frac{1}{(\bar{z}-\bar{w})^2} + \frac{\partial_{\bar{w}}}{(\bar{z}-\bar{w})}]\hat{\bar{J}}_{n(r)\mu}(\bar{w}) + \quad \text{(A.3b)}$$
$$+\frac{\mathcal{N}_{n(r)\mu}{}^{n(s)\nu;n(r)-n(s),\delta}(\sigma) : \hat{\bar{J}}_{n(s)\nu}(\bar{w})\hat{\bar{J}}_{n(r)-n(s),\delta}(\bar{w}) :}{\bar{z}-\bar{w}} + \mathcal{O}(\bar{z}-\bar{w})^0\,,$$

where (A.3a) and the twisted tensors $\mathcal{M}(\sigma)$ and $\mathcal{N}(\sigma)$ are given as duality transformations in [32]. For the special case of WZW orbifolds $A_g(H)/H$, one finds [32]

$$\mathcal{M}_{n(r)\mu}{}^{n(s)\nu}(\sigma) = \delta_{n(r)\mu}{}^{n(s)\nu}, \quad \mathcal{N}_{n(r)\mu}{}^{n(s)\nu;n(t)\delta}(\sigma) = 0, \qquad \sigma = 0, \ldots, N_c - 1 \quad \text{(A.4)}$$

so eq. (A.3) reduces to the results given in (5.11).

We finally comment on the general permutation orbifold $A(H)/H$, H (permutation)$\subset$ Aut(g). In this case, we may use (A.2) and the ground state conditions (7.9) to compute the ground state conformal weights (conformal weights of the twist fields)

$$L_\sigma(m \geq 0)|0\rangle_\sigma = \delta_{m,0}\hat{\Delta}_0(\sigma)|0\rangle_\sigma, \qquad \bar{L}_\sigma(m \geq 0)|0\rangle_\sigma = \delta_{m,0}\hat{\bar{\Delta}}_0(\sigma)|0\rangle_\sigma \quad \text{(A.5a)}$$

$$\begin{aligned}\hat{\bar{\Delta}}_0(\sigma) &= \mathcal{L}^{n(r)aj;-n(r),bl}(\sigma)\,\frac{\overline{-n(r)}}{2\rho(\sigma)}\left(1 - \frac{\overline{-n(r)}}{\rho(\sigma)}\right)\mathcal{G}_{n(r)aj,-n(r),bl}(\sigma)\\ &= \mathcal{L}^{n(r)aj;-n(r),bl}(\sigma)\,\frac{\bar{n}(r)}{2\rho(\sigma)}\left(1 - \frac{\bar{n}(r)}{\rho(\sigma)}\right)\mathcal{G}_{n(r)aj;-n(r),bl}(\sigma) = \hat{\Delta}_0(\sigma)\end{aligned} \quad \text{(A.5b)}$$

where we have also used eq. (6.8e) to complete (A.5b). A more explicit expression for $\hat{\Delta}_0(\sigma)$ is given in ref. [32]. The equality (A.5b) of the right and left mover ground state conformal weights is a consistency check on the form (5.15) of the general twisted current algebra. We further expect (but have not yet tried to prove) that the Virasoro generators $L_\sigma(m)$, $\bar{L}_\sigma(m)$ of the general permutation orbifolds are copies when $\bar{L}_\sigma(m)$ is expressed in terms of the rectified right mover current modes in eq. (7.22).

B. The integrals $I_{\bar{n}/\rho}$

In this appendix, we evaluate the integrals $I_{\frac{\bar{n}}{\rho}}$ which are encountered for the permutation orbifolds in section 9. The integrals are defined up to additive constants as

$$I_{\frac{\bar{n}}{\rho}}(y) \simeq \int^y \frac{dx}{x-1}x^{-\frac{\bar{n}}{\rho}} \simeq \rho\int^{y^{1/\rho}}\frac{dz}{z^\rho - 1}z^{\rho-1-\bar{n}} \simeq \rho\int^{y^{-1/\rho}}\frac{dz}{z^\rho-1}z^{\bar{n}-1} \quad \text{(B.1a)}$$

$$\partial_1 I_{\frac{\bar{n}}{\rho}}\left(\frac{z_1}{z_2}\right) = \frac{1}{z_{12}}\left(\frac{z_2}{z_1}\right)^{\frac{\bar{n}}{\rho}}, \qquad \partial_2 I_{\frac{\bar{n}}{\rho}}\left(\frac{z_1}{z_2}\right) = \frac{1}{z_{21}}\left(\frac{z_1}{z_2}\right)^{1-\frac{\bar{n}}{\rho}}, \qquad I_{\frac{\bar{n}}{\rho}}(y) \simeq I_{\frac{\rho-\bar{n}}{\rho}}(y^{-1}), \quad \text{(B.1b)}$$

where $\bar{n} \in \{0, \cdots, \rho - 1\}$ and the symmetry relation in (B.1b) follows by comparing the last two integral representations in (B.1a).

The z integrals can be simplified by the identities

$$z^\rho - 1 = \prod_{r=0}^{\rho-1}(z - z_{r/\rho}), \qquad z_{r/\rho} \equiv e^{2\pi i \frac{r}{\rho}}\,, \tag{B.2a}$$

$$\frac{1}{z^\rho - 1} = \sum_{r=0}^{\rho-1} \frac{a_r(\rho)}{z - z_{r/\rho}}, \qquad a_r(\rho) = \left(\prod_{s\neq r}(z_{r/\rho} - z_{s/\rho})\right)^{-1} \tag{B.2b}$$

so that we obtain for the first z integral

$$I_{\frac{\bar{n}}{\rho}}(y) \simeq \rho \sum_{r=0}^{\rho-1} a_r(\rho) \int^{y^{1/\rho}-z_{r/\rho}} \frac{du}{u}(u + z_{r/\rho})^{\rho-1-\bar{n}} \tag{B.3}$$

after a change of variable to u. This can be evaluated via the binomial theorem, with the result

$$I_{\frac{\bar{n}}{\rho}}(y) \simeq \rho \sum_{r=0}^{\rho-1} a_r(\rho)(z_{r/\rho})^{\rho-1-\bar{n}} \left[\ln(y^{\frac{1}{\rho}} - z_{r/\rho}) + \sum_{l=1}^{\rho-1-\bar{n}} \frac{1}{l}\binom{\rho-1-\bar{n}}{l}(y^{\frac{1}{\rho}} z_{r/\rho}^{-1} - 1)^l\right]. \tag{B.4}$$

This form is simplest for $\bar{n}$ near $\rho - 1$, and the symmetry relation in (B.1b) gives the corresponding results for low $\bar{n}$. As examples we list the results

$$e^{I_0(y)} \propto y - 1, \qquad e^{I_{\frac{1}{2}}(y)} \propto e^{I_{\frac{1}{2}}(y^{-1})} \propto \frac{y^{\frac{1}{2}} - 1}{y^{\frac{1}{2}} + 1} \tag{B.5a}$$

$$e^{I_{\frac{\rho-1}{\rho}}(y)} \propto e^{I_{\frac{1}{\rho}}(y^{-1})} \propto \prod_{r=0}^{\rho-1}(y^{\frac{1}{\rho}} - z_{r/\rho})^{\rho a_r(\rho)} \tag{B.5b}$$

$$e^{I_{\frac{\rho-2}{\rho}}(y)} \propto e^{I_{\frac{2}{\rho}}(y^{-1})} \propto \prod_{r=0}^{\rho-1}(y^{\frac{1}{\rho}} - z_{r/\rho})^{\rho a_r(\rho) z_{r/\rho}}\ e^{\rho a_r(\rho)(y^{1/\rho} - z_{r/\rho})} \tag{B.5c}$$

which are given up to multiplicative constants.

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